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OPTIMAL MEASUREMENTS

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Abstract - The mathematical justification of the algorithm for numerical restoration of the signal distorted by inertia and resonances in measuring transducer is presented. The structure of this algorithm is described. As an example, the numerical restoration of the test signal is shown. One of the directions for development of this theory is mathematical justification of restoration of stochastic signals, in particular "white noise".

Keywords: Leontief type system, optimal measurement, dynamically distorted signal, measuring transducer, additive "noise"

1. INTRODUCTION

The optimal measurement theory arose as a branch of the inverse problems theory [1]. Another approach for solution of optimal measurement theory problems is suggested in [2]. It is based on system of equations

$$\dot{x} = Ax + Bu, y = Cx + Du, \tag{1}$$

taken from the remote control theory. System (1) is solved by means of transfer functions. Lately authors developed a new direction in mathematical theory of dynamical measurements based on ideas and methods of Sobolev type equations theory [3]. First results in this direction are presented in [4].

Our theory is based on a mathematical model (MM) of measuring transducer (MT), presented by a system of the Leontief type equations

$$L\dot{x} = Mx + Du, \tag{2}$$

where L and M are square matrices of order n, characterizing the construction of MT, and it is possible that $detL = 0$. A vector-function $x = col(x_1, x_2, \ldots, x_n)$, $x_k = x_k(t)$, $t \in [0, \tau]$, $k = 1, 2, \ldots, n$, corresponds to the state of MT, a square matrix D of order n determines the conditions for measuring of the input signal $u = col(u_1, u_2, \ldots, u_n)$. Due to the discreteness of L-spectrum of matrix M system of equations (2) can be endowed with the Showalter – Sidorov initial condition

$$[(\alpha L - M)^{-1}L]^{p+1}(x(0) - x_0) = 0, \tag{3}$$

which is equivalent to the Cauchy initial condition $x(0) = x_0$ in the case $detL \neq 0$. Here $\alpha \in \rho^L(M)$, where $\rho^L(M)$ is an L-resolvent set of matrix M. (Definition of L-spectrum $\sigma^L(M)$ of matrix M, $\rho^L(M)$ and meaning of number $p \in \{0\} \cup \mathbb{N}$ will be given below.) Justification of reduction $(1) \to (2)$ can be found in [5]. Description of the Showalter – Sidorov condition see in [6].

To find an optimal measurement, it is necessary to introduce the space of measurements $\mathfrak{U} = \left\{u \in L_2((0,\tau);\mathbb{R}^n) : u^{(p+1)} \in L_2((0,\tau);\mathbb{R}^n)\right\}$. Next, in the space $\mathfrak{U}$ we need to allocate a closed and convex set of admissible measurements $\mathfrak{U}_{a\partial} \subset \mathfrak{U}$. The set $\mathfrak{U}_{a\partial}$ accumulates a priori information about input signals. Such information must exist, since the first commandment of the Metrology says that "it is impossible to measure the unknown". On the set $\mathfrak{U}_{a\partial}$ define the penalty functional

$$J(u) = \alpha \sum_{k=0}^{1} \int_0^\tau \left|\left| y^{(k)}(t) - y_0^{(k)}(t) \right|\right|^2 dt + $$
$$+ \beta \sum_{k=0}^{K} \int_0^\tau \left\langle N_k u^{(k)}(t), u^{(k)}(t) \right\rangle dt. \tag{4}$$

Here $y = col(y_1, y_2, \ldots, y_n)$ is a vector-function, $y_k = y_k(t)$, $t \in [0, \tau]$, $k = 1, 2, \ldots, n$, corresponding to the output signal. This vector-function $y = y(t)$ can be found using the state $x = x(t)$ of MT by formula

$$y = Cx, \tag{5}$$

where C is a square matrix of order n; $y_0 = y_0(t)$, $t \in [0, \tau]$ is an output signal taken from the real MT; N_k, $k = 0, 1, \ldots, K$, $K \leq p+1$ are symmetric nonnegatively defined matrices of order n, modelling filters that "calm" resonances in chains of MT; $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are Euclidean norm and inner product in $\mathbb{R}^n$ respectively; $\alpha, \beta \in \mathbb{R}_+$, $\alpha + \beta = 1$ are normalizing multipliers. The minimum point of functional (4) on the set $\mathfrak{U}_{a\partial}$ is understood as *an optimal measurement*.

The algorithm for numerical solution of (2) – (5) (i.e. the algorithm for finding the approximate optimal measurement) is based on ideas and methods of optimal control theory for Leontief type systems [7] that was used for solving of economical problems. Firstly this theory was used for finding of the optimal measurement only for MT with inertia [8]. Subsequently, after a thorough analysis [9] we managed to extend it to the case of MT with resonances in its chains. In Sec. 3 we briefly outline our method of finding the approximate optimal measurement providing it with results of one of test computational experiments. Details can be found in [10].

One of possible directions of development of our theory is consideration of MT with noise. By now MM with additive noise, i.e. systems of the form (2) where the random process $\varphi = \varphi(t)$ stands instead of vector-function $u = u(t)$, are studied rather well. The case when φ is a white noise, i.e. a generalized derivative of the Wiener process, is the most interesting. The main difficulty is that the system of the form (2) is solved not only by integration but also by differentiation. Our approach is based on the conception of "white noise" [11] where the Nelson – Gliklikh derivative is used instead of a generalized derivative. In Sec. 4 we give some sketches of this theory, details can be found in [12].

2. MATHEMATICAL BASEMENT

Let L and M be square matrices of order n. Consider the *L-resolvent set* $\rho^L(M) = \{\mu \in \mathbb{C} : det(\mu L - M) \neq 0\}$ and the *L-spectrum* $\sigma^L(M) = \mathbb{C}\backslash\rho^L(M)$ of the matrix M. Obviously, the L-resolvent set $\rho^L(M) = \rho(L^{-1}M) = \rho(ML^{-1})$ if $detL \neq 0$. Further, the L-resolvent set $\rho^L(M) = \emptyset$ if $kerL \cap kerM \neq \{0\}$. Define the matrix M *to be regular with respect to the matrix* L (briefly, *L-regular*), if the L-spectrum of matrix M is bounded (in particular, *the set* $\sigma^L(M) = \emptyset$ for $M = \mathbb{I}_n$, and L being a nilpotent matrix). Note that the term "L-regular matrix M" is equivalent to the term "regular matrix pencil $\mu L - M$" in the sense of K. Weierstrass (cited by [13], Ch. 12). This term appeared similarly to the term "(L, σ)-bounded operator M" (see, for example, [3], Ch. 5).

In the case of L-regularity of matrix M by the Weierstrass theorem (see, for example, [4]) there exist nondegenerate matrices F and U such that $FMU = diag\{\mathbb{I}_m, S\}$, $FLU = diag\{N_{\nu_1}, N_{\nu_2}, \dots, N_{\nu_k}, \mathbb{I}_l\}$, where N_{ν_j} are Jordan boxes of order ν_j, $j = 1, 2, \dots, k$ with zeros on main diagonals, $\mathbb{I}_l$ and $\mathbb{I}_m$ are identity matrices of order l and m respectively, $l = n - m$, $m = \sum\limits_{j=1}^{k} \nu_j$, S is a square matrix of order l. Let $p = \max\{\nu_1, \nu_2, \dots, \nu_k\}$. Add here the case when $p = 0$ (i.e. $\det L \neq 0$ or $\det L = 0$ and $FLU = diag\{\mathbb{O}_m, \mathbb{I}_l\}$). Therefore for any L-regular matrix M there exists a unique number $p \in \{0\} \cup \mathbb{N}$ defined above.

Further we need a condition $\det M \neq 0$ to be fulfilled. Note that in the case of L-regularity of matrix M it does not reduce the generality of our consideration. Indeed, making change $x(t) = e^{\alpha t}y(t)$, where $\alpha \in \rho^l(M)$, we get a system of equations $L\dot{y} = M'u + e^{-\alpha t}Du$, where $M' = M - \alpha L$ and $\det M' \neq 0$. Obviously, after solving this system one can get the solution $x = x(t)$ of initial system (2) using back change $y(t) = e^{-\alpha t}x(t)$.

Theorem 1. [4], [5] *Let L and M be square matrices of order n, and matrix M be L-regular with* $\det M \neq 0$. *Then for any $x_0 \in \mathbb{R}^n$ and $u \in C^{p+1}((0,\tau);\mathbb{R}^n) \cap C^p([0,\tau];\mathbb{R}^n)$ there exists a unique solution $x \in C^1((0,\tau);\mathbb{R}^n) \cap C([0,\tau];\mathbb{R}^n)$ given by*

$$x(t) = \lim_{k\to\infty} \left[\sum_{q=0}^{p} \left(M^{-1} \left(\left(kL_k^L(M)\right)^{p+1} - \mathbb{I}_n \right) L \right)^q M^{-1} \times \right.$$

$$\times \left(\mathbb{I}_n - \left(kL_k^L(M)\right)^{p+1} \right) (Du)^{(q)}(t) +$$

$$+ \left(\left(L - \frac{t}{k} M \right)^{-1} L \right)^k x_0 + \tag{6}$$

$$+ \int_0^t \left(\left(L - \frac{t-s}{k} M \right)^{-1} L \right)^k \left(L - \frac{t-s}{k} M \right)^{-1} \times$$

$$\left. \times \left(kL_k^L(M)\right)^{p+1} Du(s)ds \right].$$

Formula (6) gives *a classical solution* to (2), (3). However we need *a strong solution* $x \in \mathfrak{X} = \{x \in L_2((0,\tau);\mathbb{R}^n) : \dot{x} \in L_2((0,\tau);\mathbb{R}^n)\}$.

Corollary 1. [4], [5], [14]. *Let conditions of Theorem 1 be fulfilled. Then for any $x_0 \in \mathbb{R}^n$, $u \in \mathfrak{U}$ there exists a unique solution $x \in \mathfrak{X}$ given by (6).*

Theorem 2. *Let conditions of Theorem 1 be fulfilled. Then for any $x_0 \in \mathbb{R}^n$ there exists a unique $v \in \mathfrak{U}_{a\partial}$ such that*

$$J(v) = \min_{u \in \mathfrak{U}_{a\partial}} J(u). \tag{7}$$

The proof of Theorem 2 is based on the Mazur theorem and can be found in [5], [14]. The optimal measurement $v = v(t)$ is called *a precise solution* of (2) – (5).

3. NUMERICAL ALGORITHM

To construct a numerical algorithm it is convenient to take the modified functional instead of functional (4):

$$J(u) = \alpha \sum_{j=0}^{1} \int_0^\tau \left|\left| y^{(j)}(t) + \tilde{y}_0^{(j)}(t) - y_0^{(j)}(t) \right|\right|^2 dt +$$

$$+ \beta \sum_{k=0}^{K} \int_0^\tau \left\langle N_k u^{(k)}(t), u^{(k)}(t) \right\rangle dt, \tag{8}$$

which differs from (4) by summand $\tilde{y}_0^{(k)}(t)$. This vector function responds to the observation obtained on real MT without useful input signal. In other words, $\tilde{y}_0^{(k)}$ is responsible for noise caused by resonances in chains of MT. The necessity of such modernization of the functional (4) was substantiated in [8]. It was shown [10] that Theorem 2 is true even if functional (4) is substituted by functional (8). A minimum point of functional (8) on $\mathfrak{U}_{a\partial}$ is called *an optimal measurement* as well, and the solution of (2), (3), (5), (7), (8) is called a precise solution of the optimal measurement problem. We emphasize that in theoretical aspect there is no difference between these two problems, however, the second problem is more useful for the construction of the algorithm for numerical solution.

On **the first step** of this algorithm we note that the space $\mathfrak{U}$ is separable by the construction. It means that there exists a sequence of the finite-dimensional ($dim\mathfrak{U}^\ell = \ell$) subspaces $\mathfrak{U}^\ell \subset \mathfrak{U}$ monotonously exhausting the space $\mathfrak{U}$ (i.e. $\mathfrak{U}^\ell \subset \mathfrak{U}^{\ell+1}$ and $\bigcup_{\ell=1}^{\infty} \mathfrak{U}^\ell$ is densely embedded in $\mathfrak{U}$). *An approximation* $u^\ell \in \mathfrak{U}^\ell$ of the measurement u is represented in the form

$$u^\ell = col\left(\sum_{j=1}^{\ell} a_{1j}\varphi_j, \sum_{j=1}^{\ell} a_{2j}\varphi_j, \ldots, \sum_{j=1}^{\ell} a_{nj}\varphi_j\right),$$

where $\{\varphi_j\}_{j=1}^{\ell}$ is an orthonormal basis of the subspace U^ℓ, and the coefficients $a_{11}, ..., a_{1\ell}, a_{21}, ..., a_{2\ell}, ..., a_{n\ell}$ are unknown. Firstly in consideration of signals distorted only by inertia of MT we used polynomials for basis functions in $\mathfrak{U}$, i.e. $\varphi_j(t) = t^j$, and (2) was considered [8] on interval $[0, 1]$. If in addition the signal is distorted by resonances in chains of MT then it is convenient to take the Fourier basis in $\mathfrak{U}$, i.e. $\varphi_j(t) = \sin(jt)$, and consider (2) on interval $[0, \pi]$.

It is natural to assume that the resonances arising in chains of MT are the perturbations of the measurements u^ℓ, i.e. instead of u^ℓ consider

$$\tilde{u}^\ell = col(u_1^\ell + A_1 \sin\omega_1 t, u_2^\ell + A_2 \sin\omega_2 t, ..., u_m^\ell + \\ + A_m \sin\omega_m t),$$

where the resonance frequencies $\omega_1, \omega_2, ..., \omega_m$ are assumed to be known, and the amplitudes $A_1, A_2, ..., A_m$ are not. Construct an *approximate solution* $x_k^\ell = x_k^\ell(t)$ of (2), (3). This solution has the form

$$x_k^\ell(t) = \sum_{q=0}^{p} \left(M^{-1}\left(\left(kL_k^L(M)\right)^{p+1} - \mathbb{I}_n\right) L\right)^q M^{-1} \times \\ \times \left(\mathbb{I}_n - \left(kL_k^L(M)\right)^{p+1}\right) (D\tilde{u}^\ell)^{(q)}(t) + \\ + \left(\left(L - \frac{t}{k}M\right)^{-1} L\right)^k x_0 + \qquad (9) \\ + \sum_{j=0}^{J} \left(\left(L - \frac{t-s_j}{k}M\right)^{-1} L\right)^k \left(L - \frac{t-s_j}{k}M\right)^{-1} \times \\ \times \left(kL_k^L(M)\right)^{p+1} D\tilde{u}^\ell(s_j)\Delta c_j,$$

where s_j and Δc_j are nodes and weights of the Gauss quadrature formula. Note that the choice of k should be bounded below [8]. By substituting x_k^ℓ in (5) instead of x we find *an approximate observation* $y_k^\ell = y_k^\ell(t)$.

In **the second step** of the algorithm substitute the data $y_0^{(j)}$ and $\tilde{y}_0^{(j)}$, $j = 0, 1$, the approximate observation y_k^ℓ instead of y and $\tilde{u}^\ell$ instead of u in the penalty functional J (8). After the calculations in (8) we obtain a functional $J^\ell = J^\ell(\mathbf{a})$, where the vector $\mathbf{a} = col(a_{11}, ..., a_{1\ell}, a_{21}, ..., a_{2\ell}, ..., a_{m\ell}, A_1, A_2, ..., A_m)$ belongs to the space $\mathbb{R}^\ell \times \mathbb{R}^m$. Where the subspace $\mathbb{R}^m$ is called a *space of resonances amplitudes*.

Refer to the set of admissible measurements $\mathfrak{U}_{a\partial}$. Typically, in applications it is not only closed and convex, but in addition it is bounded. Let the set $\mathfrak{U}_{a\partial}$ be closed, convex and bounded then there exists a sequence of convex compacts $\{\mathfrak{U}_{a\partial}^\ell\}$, $\mathfrak{U}_{a\partial}^\ell \subset \mathfrak{U}^\ell$ monotonically exhausting the set $\mathfrak{U}^\ell$. In our considerations we can construct a convex compact set in the space $\mathbb{R}^{\ell m}$ isomorphic to $\mathfrak{U}_{a\partial}^\ell$. Further this compact set will be denoted by the same symbol $\mathfrak{U}_{a\partial}^\ell$. In the space of resonances amplitudes $\mathbb{R}^m$ choose a convex compact set $\mathfrak{U}_{a\partial}^m$ accumulating a priori information about MT resonance's amplitudes. Find the minimum of the functional J^ℓ on the set $\mathfrak{U}_\partial^\ell \times \mathfrak{U}_\partial^m$ that exists (and is unique) due to the Mazur Theorem.

Substituting the values $\tilde{a}_{11}, ..., \tilde{a}_{1\ell}, \tilde{a}_{21}, ..., \tilde{a}_{m\ell}$ of the minimum point

$$\tilde{\mathbf{a}} = col(\tilde{a}_{11}, ..., \tilde{a}_{1\ell}, \tilde{a}_{21}, ..., \tilde{a}_{2\ell}, ..., \tilde{a}_{m\ell}, \tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_m)$$

of the functional J^ℓ on the set $\mathfrak{U}_{a\partial}^\ell \times \mathfrak{U}_{a\partial}^m$ into $\mathfrak{U}^\ell$ we get the vector function

$$u_k^\ell = col\left(\sum_{j=1}^{\ell} \tilde{a}_{1j}\varphi_j, \sum_{j=1}^{\ell} \tilde{a}_{2j}\varphi_j, \ldots, \sum_{j=1}^{\ell} \tilde{a}_{mj}\varphi_j\right),$$

which is called *an approximate optimal measurement*. The superscript of u_k^ℓ defines the dependence on "approximate space" $\mathfrak{U}^\ell$, and the subscript defines the dependence on approximation (9). Note that we have simultaneously found the resonance amplitudes $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_m$ that we are not interested in as an approximate state of MT. (The state of MT can be found from (9) by substituting of the vector function

$$\tilde{u}_k^\ell = col\left(u_{k1}^\ell + \tilde{A}_1 sin\omega_1 t, u_{k2}^\ell + \tilde{A}_2 sin\omega_2 t, ..., u_{km}^\ell + \\ + \tilde{A}_m sin\omega_m t\right)$$

instead of $\tilde{u}^\ell$). Note again that, for simplicity, the time t in the algorithm belongs to $[0, \pi]$ (i.e. in (3) we assume $\tau = \pi$). To consider the other intervals it is necessary to use the correction coefficients for t in (8), (9).

Theorem 3. [10] *Let conditions of Theorem 1 be fulfilled. Then*

$$\lim_{\ell\to\infty} \lim_{k\to\infty} u_k^\ell \to v$$

in the norm of $\mathfrak{U}$.

In other words, there is a convergence of approximate optimal measurements to a precise solution of (2), (3), (5), (7), (8) obtained in Theorem 2.

During the computational experiment the test signal $u = \sin(t)$ was fed to the input of MM of MT (i.e. it was substituted into a system of equations of the form (2) [9]) and was "distorted" by resonances. Afterwards it was restored according to the suggested algorithm. The result is shown on Fig. 1.

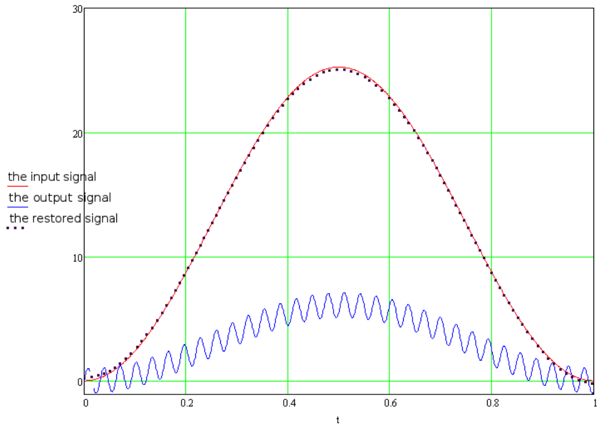

Fig. 1. The results of restoration of a test signal.

4. OPTIMAL MEASUREMENTS AND "NOISES"

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $\mathbb{R}$ be a set of real numbers endowed with Boreal σ-algebra. The measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable*. The set of random variables forms a Hilbert space $\mathbf{L_2}$ with inner product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. Let $\Im \subset \mathbb{R}$ be some interval. Consider two mappings: $f : \Im \to \mathbf{L_2}$, which maps each $t \in \Im$ to a random variable $\xi \in \mathbf{L_2}$, and $g : \mathbf{L_2} \times \Omega \to \mathbb{R}$, which maps every pair (ξ, ω) to the point $\xi(\omega) \in \mathbb{R}$. The mapping $\eta : \Im \times \Omega \to \mathbb{R}$ of the form $\eta = \eta(t, \omega) = g(f(t), \omega)$ is called a *(one-dimensional) random process*. Thus, for every fixed $t \in \Im$ the random process $\eta = \eta(t, \cdot)$ is the random variable, i.e. $\eta(t, \cdot) \in \mathbf{L_2}$, and for every fixed $\omega \in \Omega$ the random process $\eta = \eta(\cdot, \omega)$ is called *the (sample) trajectory*.

The (one-dimensional) Wiener process $\beta = \beta(t)$, modelling Brownian motion on the line in Einstein – Smoluhovsky theory, is among the most important examples of continuous Gaussian random processes.

Now fix random process η and $t \in (\varepsilon, \tau) \subset \mathbb{R}$ and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by the random variable $\eta(t)$. For the sake of brevity by $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$ denote conditional expectation with respect to $\mathcal{N}_t^\eta$. The random variable

$$D\eta\,(t, \cdot) = \lim_{\triangle t \to 0+} \mathbf{E}_t^\eta \left(\frac{\eta\,(t + \triangle t, \cdot) - \eta(t, \cdot)}{\triangle t} \right)$$

$$\left(D_*\eta\,(t, \cdot) = \lim_{\triangle t \to 0-} \mathbf{E}_t^\eta \left(\frac{\eta\,(t, \cdot) - \eta\,(t - \triangle t, \cdot)}{\triangle t} \right) \right),$$

is called *a forward* $D\eta(t, \cdot)$ *(a backward* $D_*\eta(t, \cdot)$*) mean derivative of the random process* η *at the point* $t \in (\varepsilon, \tau)$ if the limit exists in the sense of uniform metric on $\mathbb{R}$. If the random process η is forward (backward) mean differentiable on (ε, τ), then $D_S\eta = \frac{1}{2}(D + D_*)\,\eta$ is called *a symmetric mean derivative* or *a Nelson – Gliklikh derivative* for brevity and is denoted by $\overset{o}{\eta}$. By $\overset{o\,(l)}{\eta}$, $l \in \mathbb{N}$ we denote the l-th Nelson – Gliklikh derivative of the random process η.

Now fix $n \in \mathbb{N}$, take n independent random processes $\{\eta_1(t), \eta_2(t), ..., \eta_n(t)\}$ and define *n-dimensional random process* by formula

$$\Theta(t) = \sum_{j=1}^{n} \eta_j(t)e_j,$$

where e_j are basis vectors, $j = \overline{1, n}$. As an example consider n-dimensional Wiener process

$$W_n(t) = \sum_{j=1}^{n} \beta_j(t)e_j,$$

where β_j, $j = \overline{1, n}$ are independent Brownian motions.

Consider so-called stochastic Leontief type system of the form

$$L \overset{o}{\xi} = M\xi + D\varphi \tag{10}$$

modelling random changes in state $\xi = \xi(t)$ of MT under the influence of random external perturbations $\varphi = \varphi(t)$. Here $\overset{o}{\xi} = \overset{o}{\xi}(t)$ is the Nelson – Gliklikh derivative of a random process $\xi = \xi(t)$, and matrices L, M and D are the same as in (2). In particular it can be that $\varphi = \overset{o}{W}_n$, where $W_n = W_n(t)$ is an n-dimensional Wiener process. For system (10) we can also set the Showalter – Sidorov initial condition of the form (3) [3] and even consider the optimal measurement problem of the form (2) – (5), (7), see [5], [14].

To consider strong solutions $\xi = \xi(t)$ of (10) we need to introduce *the space of integrable "noises"*. Fix the interval (ε, τ) and by $\mathbf{L_2}((\varepsilon, \tau); \mathbb{R})$ denote the space of stochastic processes with any trajectory almost surely lying in $L_2((\varepsilon, \tau); \mathbb{R})$. The space $\mathbf{L_2}((\varepsilon, \tau); \mathbb{R})$ is a Hilbert space with inner product $[\eta, \xi] = \int\limits_{\varepsilon}^{\tau} \mathbf{E}\eta(t)\xi(t)dt$. Similarly construct the Hilbert space $\mathbf{L_2}((\varepsilon, \tau); \mathbb{R}^n)$ with inner product $[\eta, \xi]_n = \int\limits_{\varepsilon}^{\tau} \mathbf{E} < \eta(t), \xi(t) > dt$.

Assuming that the matrix M is (L, p)-regular, $p \in \{0\} \cup \mathbb{N}$, supply system (10) with the Showalter – Sidorov initial condition

$$\left[(\alpha L - M)^{-1} L\right]^{p+1} (\xi(0) - \xi_0) = 0. \tag{11}$$

Fix the interval $(0, \tau) \subset \mathbb{R}_+$ and construct a stochastic MT state space $\Xi = \{\xi \in \mathbf{L}_2((0, \tau); \mathbb{R}^n) : \overset{o}{\xi} \in \mathbf{L}_2((0, \tau); \mathbb{R}^n)\}$ and *stochastic measurements space* $\Phi = \{\varphi \in \mathbf{L}_2((0, \tau); \mathbb{R}^n) : \overset{o\,(p+1)}{\varphi} \in \mathbf{L}_2((0, \tau); \mathbb{R}^n\}$.

Note that if any trajectory of a random process $\overset{o\,(k+1)}{\psi} = \overset{o\,(k+1)}{\psi}(t)$, $t \in (0, \tau)$, $k \in \{0\} \cup \mathbb{N}$ lies in $\mathbf{L}_2((0, \tau); \mathbb{R}^n)$ then the same trajectory of the random process $\overset{o\,(k)}{\psi}$ is absolutely continuous on $[0, \tau]$ by the Sobolev imbedding theorems. Therefore, condition (11) and stochastic spaces Ξ, Φ are defined correctly. Fix $\varphi \in \Phi$. The random process $\xi \in \Xi$ is called *a strong solution* of system

(10), if for any trajectory of φ there exists a.s. trajectory ξ almost everywhere (a.e.) on $(0, \tau)$ satisfying (10). It is called *a strong solution of problem* (10), (11) if it satisfies condition (11) for some $\xi_0 \in \mathbf{L}_2$. Sometimes instead of condition (11) it is convenient to use the *weakened* (in the sense of S. G. Krein) *Showalter – Sidorov condition*

$$\lim_{t \to 0+} \left[(\alpha L - M)^{-1} L \right]^{p+1} (\xi(t) - \xi_0) = 0. \qquad (12)$$

For example, it is useful when $\varphi = \overset{o}{W}_n$, where $\overset{o}{W}_n = \overset{o}{W}_n\ (t)$ is the "white noise" [11]. There was proved the existence and uniqueness of a strong solution of (10), (11) as well as (10), (12).

Implement MM (10), (11) (or (10), (12)) with the system of equations

$$\eta = C\xi,$$

modelling observations of the input signal $\varphi = \varphi(t)$ and the penalty functional

$$J(\varphi) = \alpha \sum_{k=0}^{1} \int_0^\tau \mathbf{E} || \overset{o}{\eta}{}^{(k)} (t) - \overset{o}{\eta}{}_0^{(k)} (t) ||^2 dt +$$

$$+ \beta \sum_{k=0}^{K} \int_0^\tau \mathbf{E} < N_k \overset{o}{\varphi}{}^{(k)} (t), \overset{o}{\varphi}{}^{(k)} (t) > dt.$$

There was proved the existence of $\psi \in \Phi_{a\partial}$ ($\Phi_{a\partial}$ is the set of *admissible stochastic measurements*), which is called an *optimal stochastic measurement*. As above the set $\Phi_{a\partial}$ is closed and convex in the space Φ.

5. SOME REMARKS

Despite the fact that the study of additive "noise" is far from completion, new results in the theory of multiplicative "noise" based on [16] were obtained recently [15]. Moreover there was discovered a new application of the theory [7], [10] to construction mixtures [17]. Finally, to avoid the impression that we are working with "noises" just because we can't deal with "real" noises, refer to [18], where infinite-dimensional spaces of noises were constructed.

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