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REFERENCE MODEL AND ALGORITHMS FOR MULTI-STATION COORDINATE METROLOGY

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Abstract - This paper described models and algorithms for analysing data captured by coordinate metrology systems such as laser tracers, laser radar, laser trackers, theodolites and indoor GPS. Algorithms use structure-exploiting techniques to determine estimates of target locations and associated variance matrix efficiently. The calculation of the complete variance matrix enables valid estimates of uncertainties for quantities derived from the solution targets to be evaluated and provides a basis for improving the design of measurement strategies to reduce uncertainties.

Keywords: algorithm, coordinate metrology, structured matrices, uncertainty evaluation

1. Introduction

This paper describes models and algorithms to determine best estimates of target locations and associated uncertainties from measurements of angle and displacement. The algorithms apply to measurements gathered by laser interferometry (multilateration, laser tracers), theodolites, laser trackers, laser radar, indoor GPS (with some extensions) or any combination of these systems [15]. The main features of the approach are: i) the solution algorithms are based on a flexible, generic, mathematical model of the measurement systems that take into account the random and systematic effects influencing the solution estimates, ii) the estimates of the target locations are determined by a solving non-linear least squares optimisation problem weighted according to the uncertainty models for the sensor measurements; if the random and systematic effects influencing the measurement results are associated with Gaussian distributions then the least squares parameter estimates are also the maximum likelihood estimates, iii) the algorithms exploit structure in the matrix equations to maximise computational efficiency and reduce memory requirements; they also employ numerically stable procedures, iv) uncertainties associated with the target locations and associated quantities are calculated in accordance with the ISO Guide to the Expression of Uncertainties in Measurement [1]; alternatively, in a Bayesian formulation [8], the uncertainties can be regarded as those determined using a Gaussian approximation to the posterior distribution for the parameter estimates, given the measurement data, v) the model caters for calibration information supplied in terms of calibrated lengths or target locations, vi) the model can be applied to a measurement system in which a set of targets moves as a rigid body within the working volume.

2. Generic mathematical model of the sensor measurements

2.1. Notation

Below, n_X is the number of targets and n_S is the number of measuring stations. If $\boldsymbol{x}_i = (x_i, y_i, z_i)^{\mathrm{T}}$, $i \in I = \{1, \ldots, n_X\}$ is a set of point coordinates then $\boldsymbol{x}_I$ is the $3m \times 1$ vector $\boldsymbol{x}_I = (x_1, y_1, z_1, x_2, \ldots, y_{n_X}, z_{n_X})^{\mathrm{T}}$. Note, I is also used to denote the identity matrix, the diagonal matrix with ones on the diagonal. The matrix

$$R(\boldsymbol{\alpha}) = R_z(\gamma)R_y(\beta)R_x(\alpha),$$

is rotation matrix constructed from plane rotations about the x-, y- and z-axes by angles α, β and γ, respectively. A fixed rotation matrix is denoted by R_0.

2.2. Model predictions for the sensor measurements

The paragraphs below follows the general scheme of relating an observation u_i to the predicted sensor response $u_i^*(\boldsymbol{a})$ based on model parameters $\boldsymbol{a}$.

Suppose a measuring station is located at $\boldsymbol{p}_k$ and a target is located at $\boldsymbol{x}_j$. The distance from station to target is $\|\boldsymbol{x}_j - \boldsymbol{p}_k\|$. It is assumed that the displacement sensor provides an estimate d_i of d_i^* where

$$d_i^* = d_i^*(\boldsymbol{x}_j, \boldsymbol{p}_k, \lambda_l, \mu_r) = (1 + \mu_r)\|\boldsymbol{x}_j - \boldsymbol{p}_k\| + \lambda_l, \quad (1)$$

where λ_l is a displacement offset and μ_r a scale correction. For example, λ_l could be the laser deadpath offset and μ_r a scale correction due to refractive index effects.

Suppose a measuring station located at $\boldsymbol{p}_k$ records azimuth and elevation angles $\boldsymbol{\theta}_i = (\theta_i, \phi_i)$ associated with a target $\boldsymbol{x}_j$. In the coordinate system associated with the target and station locations, the unit normal from $\boldsymbol{p}_j$ to $\boldsymbol{x}_j$ is

$$\boldsymbol{n}^*(\boldsymbol{x}_j, \boldsymbol{p}_k) = \frac{\boldsymbol{x}_j - \boldsymbol{p}_k}{\|\boldsymbol{x}_j - \boldsymbol{p}_k\|}. \quad (2)$$

In the coordinate system of the measuring station located at $\boldsymbol{p}_k$, the line of sight vector $\boldsymbol{v}(\theta_i^*, \phi_i^*)$ of the target $\boldsymbol{x}_j$ is defined in terms of horizontal and vertical (azimuth and elevation) angles (θ_i^*, ϕ_i^*) by

$$\boldsymbol{v}(\theta_i^*, \phi_i^*) = (\cos\theta_i^* \cos\phi_i^*, \sin\theta_i^* \cos\phi_i^*, \sin\phi_i^*)^{\mathrm{T}}. \quad (3)$$

The two frames of references are related through the rotation matrix

$$R_k(\boldsymbol{\alpha}_k) = R(\boldsymbol{\alpha}_k)R_{0,k},$$

so that equating $\boldsymbol{v}(\theta_i^*, \phi_i^*)$ with

$$\boldsymbol{c}_{jk}^* = (c_{x,jk}^*, c_{y,jk}^*, c_{z,jk}^*)^{\mathrm{T}} = R_k(\boldsymbol{\alpha}_k)\boldsymbol{n}^*(\boldsymbol{x}_j, \boldsymbol{p}_k), \quad (4)$$

the exact angles $\theta_i^* = \theta^*(\boldsymbol{x}_j, \boldsymbol{p}_k, \boldsymbol{\alpha}_k)$ and $\phi_i^* = \phi_i^*(\boldsymbol{x}_j, \boldsymbol{p}_k, \boldsymbol{\alpha}_k)$ are defined as functions of $\boldsymbol{x}_j$, $\boldsymbol{p}_k$ and $\boldsymbol{\alpha}_k$ by

$$\theta_i^* = \tan^{-1}\left(\frac{c_{y,jk}^*}{c_{x,jk}^*}\right), \quad \phi_i^* = \sin^{-1} c_{z,jk}^*. \quad (5)$$

2.3. Target and configuration parameters

It is convenient to differentiate between the parameters $\boldsymbol{p}_k$, $\boldsymbol{\alpha}_k$, λ_l and μ_r, that relate to the configuration of the system and systematic effects common to a number of measurements, and the target location parameters $\boldsymbol{x}_j$. We let $\boldsymbol{b}$ denote the vector of all the configuration parameters and $\boldsymbol{x}_I$ the vector of all the target parameters. We also write

$$d_i^* = d_i^*(\boldsymbol{x}_j, \boldsymbol{b}), \quad \theta_i^* = \theta_i^*(\boldsymbol{x}_j, \boldsymbol{b}), \quad \phi_i^* = \phi_i^*(\boldsymbol{x}_j, \boldsymbol{b}),$$

emphasising that a model prediction depends on the configuration parameters $\boldsymbol{b}$ but only on one set of target parameters $\boldsymbol{x}_j$.

2.4. Observation equations

It is assumed that the observation d of distance is modelled as $d = d^* + \epsilon_D$, where ϵ_D is a realisation of a random variable with expectation zero and variance σ_D^2. We assume that for each measurement, σ_D depends on two statistical parameters $\sigma_{D,A}$ and $\sigma_{D,R}$, with

$$\sigma_D^2 = \sigma_{D,A}^2 + \sigma_{D,R}^2\|\boldsymbol{x} - \boldsymbol{p}\|^2,$$

so that the uncertainty associated with the displacement measurement, potentially, has a component dependent on the distance $\|\boldsymbol{x} - \boldsymbol{p}\|$ from the station to the target. The length dependence could arise from random fluctuations in refractive index, or random effects associated with laser frequency, for example.

For azimuth angle observations, the statistical characterisation has the form $\theta = \theta^* + \epsilon_A$, where ϵ_A is a realization of a random variable with expectation zero and variance σ_A^2. We assume that for each measurement, σ_A depends on two statistical parameters $\sigma_{A,A}$ and $\sigma_{A,R}$, with

$$\sigma_A^2\|\boldsymbol{x} - \boldsymbol{p}\|^2 \cos^2\phi^* = \sigma_{A,A}^2 + \sigma_{A,R}^2\|\boldsymbol{x} - \boldsymbol{p}\|^2.$$

The term $\sigma_{A,A}$ accounts for random effects associated with the target that will have a larger impact on the angle measurements for targets closer to the station. The dependence on elevation angle ϕ^* models the fact that as the targets move away from the equatorial plane, the azimuth angle is less well defined.

Similarly for elevation angle observations, the statistical characterisation has the form $\phi = \phi^* + \epsilon_E$, where ϵ_E is a realisation of a random variable with expectation zero and variance σ_E^2. We assume that for each measurement, σ_E depends on two statistical parameters $\sigma_{E,A}$ and $\sigma_{E,R}$, with

$$\sigma_E^2\|\boldsymbol{x} - \boldsymbol{p}\|^2 = \sigma_{E,A}^2 + \sigma_{E,R}^2\|\boldsymbol{x} - \boldsymbol{p}\|^2.$$

We write the generic observation equation in the form

$$u_i = u_i^*(\boldsymbol{x}_j, \boldsymbol{b}) + \epsilon_i, \quad (6)$$

where u_i is the observed sensor reading, u_i^* is the model prediction for the sensor reading, and ϵ_i is a sample from a statistical distribution with mean zero and standard deviation σ_i. Associated to each i are indices $j(i)$, $k(i)$, etc., specifying the target, station, etc., involved in the ith measurement.

2.5. Calibration information

Additional information about the targets and/or configuration parameters may be available, for example, an estimate of the distance $\|\boldsymbol{z}_j - \boldsymbol{z}_q\|$ between two targets $\boldsymbol{z}_j$ and $\boldsymbol{z}_q$. Other observational information about the configuration parameters may be supplied, e.g., relating to the offset parameters λ_l and μ_r. If we expand the vector $\boldsymbol{b}$ of configuration parameters to include any targets $\boldsymbol{z}_j$ involved in calibration information, the calibration information can usually be written as

$$\boldsymbol{h}_0 = \boldsymbol{h}(\boldsymbol{b}) + \boldsymbol{\delta}_{\boldsymbol{h}}, \quad (7)$$

where $\boldsymbol{h}_0$ is the vector of calibrated values associated with functions $\boldsymbol{h}(\boldsymbol{b})$ of the configuration parameters, and $\boldsymbol{\delta}_{\boldsymbol{h}}$ are random effects associated with a multivariate distribution with mean $\mathbf{0}$ and variance matrix $V_{\boldsymbol{h}}$.

2.6. Frame of reference constraints

In order to define a unique frame of reference for the target and station locations, it is usually necessary to apply frame constraints to a subset of the parameters $\boldsymbol{x}_I$ and $\boldsymbol{b}$. We assume that these constraints are applied as linear equality constraints on the configuration parameters $\boldsymbol{b}$ of the form

$$C^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{c}_0. \quad (8)$$

Usually the constraints apply only to the parameters $\boldsymbol{p}_k$ and, if angle measurements are involved, $\boldsymbol{\alpha}_k$.

We can use orthogonal projections to eliminate these constraints. Suppose there are n_0 configuration parameters and $n_C < n_0$ linear constraints. We assume that C is of full column rank and has the QR factorisation [10]

$$C = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} S \\ 0 \end{bmatrix}, \quad (9)$$

where V_1 and V_2 are the first n_C and last $n_0 - n_C$ columns of the orthogonal factor V. If $\boldsymbol{b}_0$ is a solution of $C^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{c}_0$, then for any $(n_0 - n_C)$-vector $\tilde{\boldsymbol{b}}$, $\boldsymbol{b} = \boldsymbol{b}_0 + V_2\tilde{\boldsymbol{b}}$ automatically satisfies the constraints and the model can be described in terms of the reduced set of parameters $\tilde{\boldsymbol{b}}$, along the fixed vector $\boldsymbol{b}_0$ and the fixed matrix V_2.

There are a number of ways that a frame of reference can be fixed. If there are three measuring stations, we can assign the location of one to the origin, that of a second to the x-axis and the third to a point on the xy-plane. Alternatively, if a measuring station involves rotation angles $\boldsymbol{\alpha}$ as well as location parameters $\boldsymbol{p}$, we can fix the reference frame by fixing $\boldsymbol{\alpha} = \boldsymbol{p} = \boldsymbol{0}$. For these schemes, a choice has to be made as to which of the stations are to be constrained. We describe below a scheme in which all measuring stations are treated on an equal basis [7].

Suppose the nominal or approximate position of a measuring station is specified by location $\boldsymbol{p}_{k,0} = (p_{x,k,0}, p_{y,k,0}, p_{z,k,0})^{\mathrm{T}}$, and orientation matrix $R_k(\boldsymbol{\alpha}_{k,0}) = R(\boldsymbol{\alpha}_{k,0})R_{0,k}$. We assume that $R(\boldsymbol{\alpha}_{k_0})$ is incorporated into $R_{0,k}$ so that $\boldsymbol{\alpha}_{k,0} = \boldsymbol{0}$. Define the 6×6 matrix C_k by

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 & -p_{z,k,0} & p_{y,k,0} \\ 0 & 1 & 0 & p_{z,k,0} & 0 & -p_{x,k,0} \\ 0 & 0 & 1 & -p_{y,k,0} & p_{x,k,0} & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad (10)$$

and set C to be the 6-column matrix with C_k in the six rows corresponding to $\boldsymbol{p}_k$ and $\boldsymbol{\alpha}_k$ in the configuration parameter vector $\boldsymbol{b}$ and zeros elsewhere. This constraint matrix has the effect of precluding any change in the parameter vector $\boldsymbol{b}$ that can be described by a rigid body transformation of the station locations. For systems involving only length measurements (such as multi-lateration), the frame of reference constraints can be applied with C_k defined to be the first three rows of the right hand side of (10).

3. Estimation of model parameters

3.1. Least squares adjustment/bundle adjustment

See also [12, 17]. The estimation of the model parameters is on the basis of the sensor data u_i, $i = 1, \ldots, m_U$, and the calibration information, where each item of information is appropriately weighted. For each sensor observation modelled as in (6), let

$$f_i(\boldsymbol{x}_j, \boldsymbol{b}) = w_i(u_i - u_i(\boldsymbol{x}_{j(i)}, \boldsymbol{b})), \quad w_i = 1/\sigma_i, \quad (11)$$

Here, the weights w_i reflect the uncertainties associated with the observations, as quantified by the standard deviations σ_i.

If the calibration information is summarised as in (7) with $V_{\boldsymbol{h}}$ full rank, we determine the Cholesky factor [10] $L_{\boldsymbol{h}}$ of $V_{\boldsymbol{h}}$, so that $V_{\boldsymbol{h}} = L_{\boldsymbol{h}} L_{\boldsymbol{h}}^{\mathrm{T}}$, and set

$$\boldsymbol{f}_0(\boldsymbol{b}) = L_{\boldsymbol{h}}^{-1}(\boldsymbol{h}_0 - \boldsymbol{h}(\boldsymbol{b})). \quad (12)$$

Here the matrix $L_{\boldsymbol{h}}^{-1}$ plays the same role as w_i in (11).

Estimates of the model parameters are found by solving a non-linear least squares problem of the form

$$\min_{\boldsymbol{x}_I, \boldsymbol{b}} \left\{ F(\boldsymbol{x}_I, \boldsymbol{b}) + F_C(\boldsymbol{b}) \right\}, \quad (13)$$

subject to $C^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{c}_0$, where $F(\boldsymbol{x}_I, \boldsymbol{b}) = \sum_{i=1}^{m_U} f_i^2(\boldsymbol{x}_{j(i)}, \boldsymbol{b})$ represents the observations involving individual targets, with f_i defined as in (11), and $F_C(\boldsymbol{b}) = \boldsymbol{f}_0^{\mathrm{T}}(\boldsymbol{b})\boldsymbol{f}_0(\boldsymbol{b})$ is the contribution associated with the calibration information for the configuration parameters, with $\boldsymbol{f}_0(\boldsymbol{b})$ defined as in (12).

3.2. Gauss-Newton algorithm

Non-linear least squares problems are generally solved using some variant of the Gauss-Newton algorithm [9] which we now briefly describe. Suppose we wish to minimise

$$F(\boldsymbol{a}) = \sum_{i=1}^{m} f_i^2(\boldsymbol{a}) \quad (14)$$

with respect to parameters $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathrm{T}}$ where $m \geq n$. If J is the associated Jacobian matrix defined at an estimate $\boldsymbol{a}$ of the solution parameters by $J_{ij} = \partial f_i / \partial a_j$, then an updated estimate is given by $\boldsymbol{a} + \boldsymbol{p}$ where $\boldsymbol{p}$ solves the matrix equation $J\boldsymbol{p} = -\boldsymbol{f}$ in the least squares sense. A stable method of solving this system (see [10], for example) is to find an orthogonal factorisation $J = QU$, where Q is an $m \times n$ orthogonal matrix and U is $n \times n$ upper triangular. The solution $\boldsymbol{p}$ can be found efficiently by solving the upper-triangular system $U\boldsymbol{p} = -Q^{\mathrm{T}}\boldsymbol{f}$.

Linear equality constraints of the form $C^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{c}_0$ can be incorporated easily as described in section 2.6. We note that if J is the Jacobian matrix associated with $\min_{\boldsymbol{a}} \sum_i f_i^2(\boldsymbol{a})$, then the reduced Jacobian matrix $\tilde{J}$ associated with $\min_{\tilde{\boldsymbol{a}}} \sum_i f_i^2(\boldsymbol{a}_0 + V_2\tilde{\boldsymbol{a}})$ is given simply by $\tilde{J} = JV_2$.

3.3. Uncertainty matrix associated with the fitted parameters

If $m \geq n$, the uncertainty matrix $V_{\boldsymbol{a}}$ associated with the fitted parameters can be estimated by

$$V_{\boldsymbol{a}} = \left(J^{\mathrm{T}}J\right)^{-1} = \left(U^{\mathrm{T}}U\right)^{-1}, \quad (15)$$

where J is the Jacobian matrix at the solution, U its upper-triangular factor. This estimate of $V_{\boldsymbol{a}}$ assumes that the observation equations have been weighted appropriately. If $m > n$, a simple posterior adjustment of the input uncertainties can be determined from $\hat{\sigma}^2 = \frac{1}{m-n}\sum_{i=1}^{m} f_i^2$, and an estimate of the uncertainty matrix using this adjusting factor is given by$\hat{V_{\boldsymbol{a}}} = \hat{\sigma}^2 \left(J^{\mathrm{T}}J\right)^{-1} = \hat{\sigma}^2 \left(U^{\mathrm{T}}U\right)^{-1}$. A more general posterior weighting scheme that can be used to adjust the weighting of the distance information relative to the angle information, for example, is discussed in [5].

If linear constraints are implemented through $\boldsymbol{a} = \boldsymbol{a}_0 + V_2\tilde{\boldsymbol{a}}$ as in section 3.2, $V_{\tilde{\boldsymbol{a}}} = \left(\tilde{J}^{\mathrm{T}}\tilde{J}\right)^{-1} = \left(V_2^{\mathrm{T}}J^{\mathrm{T}}JV_2\right)^{-1}$, and $V_{\boldsymbol{a}} = V_2 V_{\tilde{\boldsymbol{a}}} V_2^{\mathrm{T}}$.

3.4. Structure-exploiting solution algorithm

The optimisation problem (13) is such that each observation equation involves $\boldsymbol{b}$ and at most one set of parameters $\boldsymbol{x}_j$. The upper-triangular factor U of the Jacobian matrix has the form

$$U = \begin{bmatrix} U_1 & & & & U_{1,0} \\ & U_2 & & & U_{2,0} \\ & & \ddots & & \vdots \\ & & & U_{n_X} & U_{n_X,0} \\ & & & & U_0 \end{bmatrix}, \quad (16)$$

and can be determined using efficient updating strategies [2, 4]. Alternatively, sparse matrix routines such as the LSQR algorithm [14] within a Gauss-Newton framework [3] can be used to determine estimates of the model parameters efficiently.

4. EFFICIENT CALCULATION OF UNCERTAINTIES ASSOCIATED WITH THE FITTED PARAMETERS

Uncertainties associated with the target and configuration parameters can be calculated following the general approach outlined in section 3.3. However the structure of the triangular factor U in (16) can be exploited to make the calculations more efficient. The $3n_X \times 3n_X$ variance matrix V_I associated with the target estimates $\{\boldsymbol{x}_j\}$, $j \in I = \{1, \ldots, n_X\}$, can be calculated as follows. The upper-triangular factor U and its inverse $S = U^{-1}$ can be partitioned as

$$U = \begin{bmatrix} U_{11} & U_{12} \\ \mathbf{0} & U_{22} \end{bmatrix}, \quad S = U^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ \mathbf{0} & S_{22} \end{bmatrix} \quad (17)$$

with $S_{11} = U_{11}^{-1}$, $S_{22} = U_{22}^{-1}$ and $S_{12} = -U_{11}^{-1}U_{12}U_{22}^{-1}$. Also

$$(U^{\mathrm{T}}U)^{-1} = \begin{bmatrix} S_{11}S_{11}^{\mathrm{T}} + S_{12}S_{12}^{\mathrm{T}} & S_{12}S_{22}^{\mathrm{T}} \\ S_{22}S_{12}^{\mathrm{T}} & S_{22}S_{22}^{\mathrm{T}} \end{bmatrix}.$$

For U in (16) where U_{11} is the $3n_X \times 3n_X$ block-diagonal matrix specified by $\{U_j\}$, U_{12} is the $3n_X \times (n_0 - n_C)$ border matrix containing $\{U_{j,0}\}$ and $U_{22} = U_0$, the inverse matrix S has exactly the same structure. In particular, the matrices S_{11} and S_{12} can be computed efficiently and the variance matrix V_I is given by $V_I = S_{11}S_{11}^{\mathrm{T}} + S_{12}S_{12}^{\mathrm{T}}$, involving the block-diagonal matrix S_{11} and border matrix S_{12}. Hence, although V_I is a full matrix, it can be stored compactly in terms of S_{11} and S_{12}. The matrix S_{11} determines the uncertainty component arising from the random effects while S_{12} determines the uncertainty component arising from uncertainties associated with the configuration parameters.

4.1. Uncertainties associated with inter-target distances

Suppose we wish to calculate the standard uncertainty $u(d_{ij})$ in the inter-target distance $d_{ij} = \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$. Let $\boldsymbol{g}_{ij}$ solve the triangular system

$$\begin{bmatrix} U_i & & U_{i,0} \\ & U_j & U_{j,0} \\ & & U_0 \end{bmatrix}^{\mathrm{T}} \boldsymbol{g}_{ij} = \hat{\sigma} \begin{bmatrix} \boldsymbol{n}_{ij} \\ -\boldsymbol{n}_{ij} \\ \mathbf{0} \end{bmatrix}, \quad (18)$$

where $\boldsymbol{n}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j / \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$. Then $u(d_{ij}) = \|\boldsymbol{g}_{ij}\|$.

4.2. Uncertainties associated with inter-target angles

Given three targets $\boldsymbol{x}_i$, $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$, the angle between the vectors $\boldsymbol{x}_j - \boldsymbol{x}_i$ and $\boldsymbol{x}_k - \boldsymbol{x}_i$ is given by

$$\alpha_{ijk} = \cos^{-1}\left(\frac{(\boldsymbol{x}_j - \boldsymbol{x}_i)^{\mathrm{T}}(\boldsymbol{x}_k - \boldsymbol{x}_i)}{d_{ij}d_{ik}}\right)$$

The evaluation of the uncertainty associated with α_{ijk} involves the submatrix

$$\begin{bmatrix} U_i & & & U_{i,0} \\ & U_j & & U_{j,0} \\ & & U_k & U_{k,0} \\ & & & U_0 \end{bmatrix}.$$

4.3. Uncertainties associated with position and shape

Suppose V is a $3m \times 3m$ variance matrix associated with a $3m$-vector of coordinates $\boldsymbol{x}_I$. Let J be the $3m \times 6$ matrix with

$$J(3i-2:3i,:) = \begin{bmatrix} 1 & 0 & 0 & 0 & -z_i & y_i \\ 0 & 1 & 0 & z_i & 0 & -x_i \\ 0 & 0 & 1 & -y_i & x_i & 0 \end{bmatrix}$$

(*cf.* Eq. (8)). If J has QR factorisation $J = Q_1T_1$, where Q_1 is a $3m \times 6$ orthogonal matrix, set $P_1 = Q_1Q_1^{\mathrm{T}}$ and $P_2 = I - P_1$. The matrices P_1 and P_2 are projections and we set $V_P = P_1VP_1^{\mathrm{T}}$ and $V_S = P_2VP_2^T$. The variance matrix V_P represents the variance component associated with position of the data points $\boldsymbol{x}_I$, that is, the variance component the rigid body variation in $\boldsymbol{x}_I$, while V_S is the variance component associated with variations in the shape of the coordinate vector $\boldsymbol{x}_I$ [7]. Uncertainties associated with inter-target distances and inter-target angles are the same if calculated using V or V_S. The variance matrix V_S often gives a more useful representation of the target uncertainties as it is independent of the method of implementing the frame of reference constraints, for example. We have

$$V = (P_1+P_1)V(P_1+P_2)^{\mathrm{T}} = V_P+V_S+P_1VP_2^{\mathrm{T}}+P_2VP_1^{\mathrm{T}},$$

so that in general $V \neq V_P + V_S$, but

$$\mathrm{trace}(V) = \mathrm{trace}(V_P) + \mathrm{trace}(V_S).$$

The trace of a variance matrix (the sum of its diagonal elements) is often used as an aggregate measure of the total variance and the decomposition into position and shape variance components preserves this aggregate measure.

The shape component from a variance matrix $V = SS^{\mathrm{T}}$ structured as in (17) involves the calculation of

$$(I - Q_1Q_1^{\mathrm{T}}) \begin{bmatrix} S_{11} & S_{12} \\ & S_{22} \end{bmatrix}.$$

Although S_{11} is a block-diagonal matrix, the matrix $Q_1Q_1^{\mathrm{T}}[S_{11}^{\mathrm{T}}\ 0]^{\mathrm{T}}$ is full. However the matrix S_{11} represents the uncertainties due to random effects and is therefore unlikely to embody a significant positional component. This means that a reasonable approximation to V_S is given by $S_S S_S^{\mathrm{T}}$ where

$$S_S = \left[\begin{array}{cc} S_{11} & \tilde{S}_{12} \\ & \tilde{S}_{22} \end{array}\right], \quad \left[\begin{array}{c} \tilde{S}_{12} \\ \tilde{S}_{22} \end{array}\right] = (I - Q_1 Q_1^{\mathrm{T}}) \left[\begin{array}{c} S_{12} \\ S_{22} \end{array}\right],$$

and can be computed and stored compactly.

5. IMPLEMENTATION MODES

5.1. Post-processing

In the post-processing mode, all the measurements are gathered and the estimates of the target coordinates are determined by solving (13). This mode has the advantage the target estimates are determined on the basis of all the available information. It has the potential disadvantage that all the measurements have to be taken before target estimates are available.

5.2. Self-calibration and measurement mode

During and initial, self-calibration phase, a set of targets is measured in order to determine sufficiently accurate estimates $\boldsymbol{b}_0$ of the configuration parameters $\boldsymbol{b}$, along with associated variance matrix $V_{\boldsymbol{b}_0}$. In the second, measurement phase, sensor measurements associated with target coordinates $\boldsymbol{x}$ are processed to determine estimates of $\boldsymbol{x}$ by solving the appropriately weighted least squares problem

$$\min_{\boldsymbol{x}} \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}, \boldsymbol{b}_0)\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{b}_0), \tag{19}$$

with $\boldsymbol{b}_0$ regarded as fixed. This calculation can be made efficiently as it involves only the three target parameters $\boldsymbol{x}$. If J is the Jacobian matrix of partial derivatives of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$, evaluated at the solution, then one estimate of the variance matrix $V_{\boldsymbol{x}}$ associated with $\boldsymbol{x}$ is given by $V_{\boldsymbol{x}} = \left(J^{\mathrm{T}} J\right)^{-1}$. This matrix reflects the uncertainties associated with the estimates due to the random effects associated with the sensor measurements but ignores the contribution arising from the uncertainties associated with the configuration parameters. It may be sufficient to use this estimate if the configuration parameters are known very accurately. Otherwise, if J_0 is the Jacobian matrix of partial derivatives of $\boldsymbol{f}$ with respect to $\boldsymbol{b}$ (evaluated at $\boldsymbol{b}_0$), then the uncertainty matrix for $\boldsymbol{x}$ is more accurately estimated by

$$V_{\boldsymbol{x}} = P\left(I + J_0 V_{\boldsymbol{b}_0} J_0^{\mathrm{T}}\right) P^{\mathrm{T}}, \quad P = \left(J^{\mathrm{T}} J\right)^{-1} J^{\mathrm{T}}.$$

The common dependence of the target estimates on the configuration parameters $\boldsymbol{b}$ introduces correlation that should be taken into account, in calculating, for example, the uncertainties associated with the inter-target distances.

5.3. Sequential updating strategy

While the calculations associated with the measurement phase can be performed very efficiently, any information regarding the configuration parameters available during the measurement phase is ignored. An alternative approach is to process the measurements taken during the measurement phase target by target and update the estimates of the configuration parameters. If $\boldsymbol{b}_{j-1}$ and $V_{\boldsymbol{b}_{j-1}} = \left(U_0^{\mathrm{T}} U_0\right)^{-1}$ represent the estimate and variance matrix for the (constrained) configuration parameters when the data for the $(j-1)$th target has been processed, the estimate of the jth target along with updated estimates for $\boldsymbol{b}$ are found by minimising

$$\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_j, \boldsymbol{b}_j)\boldsymbol{f}(\boldsymbol{x}_j, \boldsymbol{b}_j) + (\boldsymbol{b}_j - \boldsymbol{b}_{j-1})^{\mathrm{T}} V_{\boldsymbol{b}_{j-1}}^{-1} (\boldsymbol{b}_j - \boldsymbol{b}_{j-1})$$

with respect to $\boldsymbol{x}_j$ and $\boldsymbol{b}_j$, a compact calculation.

At the end of the updating strategy, the estimates and associated uncertainties for the configuration parameters are, to first order, the same as those determined by the post-processing mode, and reflect all the information available from the measurements. However, only the last target estimate reflects all the data. At this stage, all the earlier target estimates can be recomputed using the final estimate of the configuration parameters, as in the measurement phase of the two-stage strategy. If this is done, then all target estimates are the same (to first order) as those determined in the post-processing mode. The sequential updating approach therefore has the advantages of providing target estimates as the measurements are targets are made and allows all the information to be used to update the target estimates, if required, once all the measurements have been made.

6. EXTENSIONS TO THE MODEL

6.1. Target assemblies

For some applications, the targets represent different positions of a fixed set of targets moving as a rigid body: $\boldsymbol{y}_{l,j} = \boldsymbol{x}_{0,j} + R^{\mathrm{T}}(\boldsymbol{\beta}_j)\boldsymbol{y}_l$. If we set $\boldsymbol{x}_j = [\boldsymbol{x}_{0,j}^{\mathrm{T}}, \boldsymbol{\beta}_j^{\mathrm{T}}]^{\mathrm{T}}$ and incorporate the parameters $\boldsymbol{y}_l$ into the vector $\boldsymbol{b}$ of configuration parameters (applying frame of reference constraints to $\boldsymbol{y}_l$, as required), estimates of $\boldsymbol{x}_j$ and hence $\boldsymbol{y}_{l,j}$ can be found by solving the equivalent of (13).

6.2. Configuration parameters incorporating instrument effects

The configuration parameters discussed here have related mainly to specifying the positions of the measuring stations. They can also be expanded to take into account instrument effects such as camera calibration parameters in photogrammetry [17] or alignment parameters in laser trackers [11, 13], for example.

6.3. Fixed monuments, floating stations

The reference model assumes that the station parameters $\boldsymbol{p}_k$ and $\boldsymbol{\alpha}_k$ are fixed over the course of the measurements. If a sufficient number fixed targets $\boldsymbol{z}_l$ are remeasured regularly, the station parameters can float so that the estimates of the targets $\boldsymbol{x}_j$ are determined relative to the fixed targets $\boldsymbol{z}_l$, with the station parameters determined as intermediate parameters. Station drift can be important for measurements of large structures lasting hours. One approach to modelling drift is as follows.

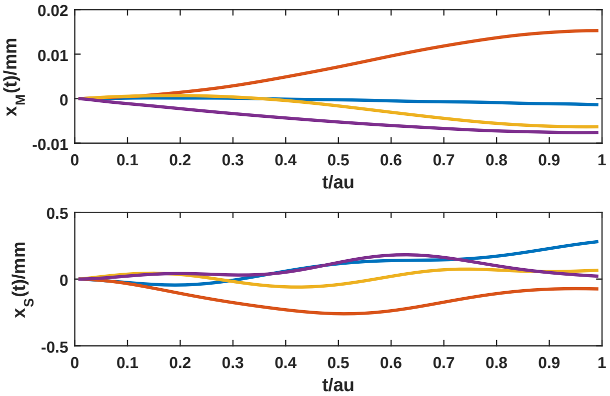

Fig. 1. Temporally correlated coordinate functions modelling drift in monuments, top, and stations, bottom.

We assume that both the monuments $\boldsymbol{z}_l = \boldsymbol{z}_l(t)$ and station parameters $\boldsymbol{p}_k = \boldsymbol{p}_k(t)$ and $\boldsymbol{\alpha}_k = \boldsymbol{\alpha}_k(t)$ are smooth functions of time t. We model these functions at temporally correlated polynomials $x(t)$ [6]. These functions are such that the covariance of $x(t)$ with $x(t')$ is given by a kernel $k(t, t'|\tau)$ so that for t' close to t, relative to a time constant τ, $x(t)$ is highly correlated with $x(t')$ [16]. We associate with the fixed monuments $\boldsymbol{z}_l(t)$ large time constants since we expect the monuments to be nearly static (in practice it is impossible to guarantee absolute stability), while the stations parameters are assigned somewhat smaller time constants depending on the stability of the environmental conditions. We also impose frame of reference constraints on the monument targets $\boldsymbol{z}_l(t)$ so that at any given time t, the frame of reference constraints are automatically satisfied by the monument locations. Figure 1 shows examples of x-coordinate functions for four monuments (top) and four stations (bottom). The top graph shows smooth and small drift with the sum of the coordinates constrained to be zero while the bottom graphs shows less smooth and larger drift with no constraint.

7. CONCLUSIONS

This paper has discussed models and algorithms for determining estimates target locations and associated uncertainties using multi-station coordinate measurement systems. The algorithms exploit the structure in the various matrices involved to determine estimates and associated uncertainties efficiently. The computational effort scales approximately linearly in the number of targets and can be used to analyse measurements relating to thousands of targets. The uncertainty evaluation algorithms take into account both systematic and random effects and lead to realistic uncertainty statements.

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