XVIII IMEKO WORLD CONGRESS
Metrology for a Sustainable Development
September, 17 – 22, 2006, Rio de Janeiro, Brazil

A SYSTEMATIC MODELLING CONCEPT FOR UNCERTAINTY ANALYSIS

Klaus-Dieter Sommer [1], *Albert Weckenmann* [2], *Bernd R. L. Siebert* [3], *Stefan Heidenblut* [1], *Karina Weißensee* [1]

[1] Landesamt fuer Mess- und Eichwesen Thueringen (LMET), D-98693 Ilmenau, Germany, klaus-dieter.sommer@lmet.de
[2] Friedrich-Alexander-University of Erlangen-Nuremberg, D-91052 Erlangen, Germany
[3] Physikalisch-Technische Bundesanstalt (PTB), D-38116 Braunschweig, Germany

Abstract: The evaluation of measurement uncertainty is based on both, the knowledge about the measuring process and the quantities which influence the measurement result. The knowledge about the measuring process is represented by the model equation which expresses the interrelation between the measurand and the input quantities. Therefore, the modeling of the measurement is a key element of modern uncertainty evaluation. A practical modeling concept has been developed that is based on the idea of the measuring chain. It gets on with only a few generic model structures. From this concept, a practical stepwise procedure has been derived.

Keywords: measurement uncertainty, measurement system, model equation.

1. INTRODUCTION

In modern uncertainty analysis [1], the knowledge about the input quantities is expressed by appropriate probability density functions (pdfs), whereas the measurement process is represented by a so-called model equation. It expresses the mathematical interrelation between the measurand Y and the input quantities $X_1,...,X_N$ as well as between their possible values, η for Y and ξ for X:

$$Y = f_M\left(X_1,...,X_N\right) \text{ and } \eta = f_M\left(\xi_1,...,\xi_N\right) \qquad (1)$$

The model equation forms the basis for the propagation of the pdfs for the input quantities and, in case of utilizing Gaussian uncertainty propagation [1], for the propagation of their expectation values and associated uncertainties respectively. Therefore, the modeling of the measurement is a key element of uncertainty evaluation. First approaches to a systematic modeling procedure for practitioners were made by Bachmair [2], Kessel [3], Kind [4] and Sommer et al. [5].

2. BASIC RELATIONSHIPS OF THE MODELLING CONCEPT

A model can serve to evaluate the original system or to draw conclusions from its behavior. In measurement techniques, usually the measurand and other influence quantities may be seen as causative signals (causes). The system transforms these causal signals into indications or output signals (effects) and, therewith, assigns values to the measurand.

In contrast to models used to describe the cause-effect behavior, often called measurement equation, for the evaluation of the measurement uncertainty an "inverse model", briefly termed "model equation" (1), is needed (see figure 1).

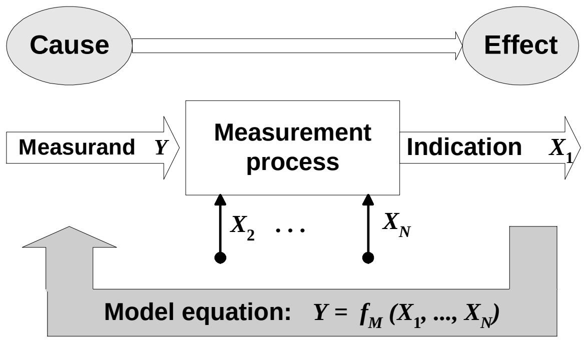


Fig. 1. - Cause-and-effect relationship and the model for uncertainty evaluation.

The modeling concept presented [5] is based on the idea of the measuring chain which constitutes the path of the measurement signal from cause to effect. It mainly refers to the ISO-GUM procedure [1] that only applies to linear or linearizable models. But basically, the concept presented is not limited to this category of models.

2.1. *Linearization*

Besides applicability to the ISO-GUM procedure [1], linear models offer some more benefits, in particular the possibility of using the well-commanded instruments of signal and system theory and, therewith, for proper modeling of time and frequency-dependent systems [6, 7].

Basically, linearization is possible if the following equation is satisfied:

$$\eta = f_M\left(\xi\right) \cong f_M\left(x\right) + \sum_{i=1}^{N} c_i\left(\xi_i - x_i\right) \qquad (2)$$

where ξ is used as an abbreviation for $\xi_1,...,\xi_N$ and x for $x_1,...,x_N$. The c_i are called sensitivity coefficients given by

$$c_i = \left. \frac{\partial f_M \left(X_1,...,X_N \right)}{\partial X_i} \right|_{\xi_i = x_i} \qquad (3)$$

and x represents the expectation values (best estimates) of the quantities X (see below).

Equation (2) will usually be satisfied if the following assumptions can be made:

- In narrow ranges around the operating points, the functional elements or steps of a measurement may be regarded to have approximately linear characteristics and can, therefore, be described by first-order Taylor series expansion. Figure 2 illustrates this linearization and the limits of its applicability.
- The (steady-state) transmission behavior of a fictitious unperturbed measuring system is related to well-known operating points given by the expectation values $\mathrm{E}[X_1],...,\mathrm{E}[X_N] = x_1,...,x_N$ of the input quantities.
- The "real world of measurement" may be taken into consideration by means of deviations of values of the real influence quantities and other parameters from the above "idealized operating conditions".

For linearization, one replaces the model equation $Y = f_M \left(X_1,...,X_N \right)$ (see equation (1)) with the relation for its values, $\eta = f_M \left(x \right) + c\left(\xi - x \right)$ (see equation (2)).

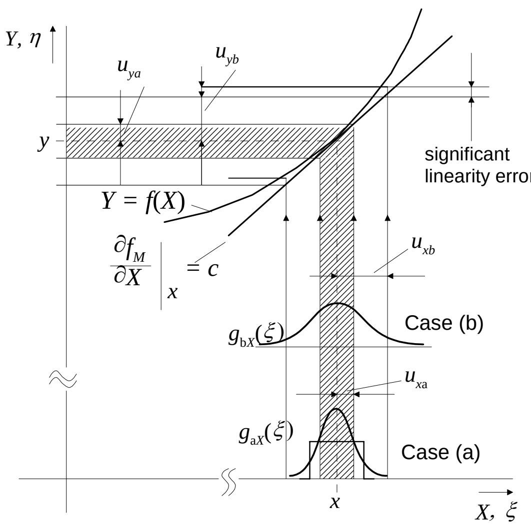


Fig. 2. - Linearization of the functional dependency of the output quantity Y. Case (a): Linear treatment is possible (equation (2) is being satisfied). Case (b): Linearization would lead to erroneous uncertainty propagation and a false value for *y*.

On the above assumptions, at least in the steady state, almost all functional elements or operational steps of a measuring system or process, may be represented by a generic "perturbed transmission element" having an approximately constant transmission factor and deviations that represent the imperfection of the measurement. Basically, this can be expressed by the following relationship:

$$X_{kOUT} = X_{kIN} \left(G_{0k} + \delta G_{0k} \right) + \delta Z_k \qquad (4)$$

where: X_{kIN} – quantity acting at the input of the element k; X_{kOUT} – quantity at the output of the element k; G_{0k} – transmission factor; δG_{0k} – parameter deviation; δZ_k – parameter deviation.

Then, for a linear or linearized model equation, the expectation value of the output quantity Y is

$$y = E\left[Y \right] = f_M \left(x_1,...,x_N \right). \qquad (5)$$

The uncertainty associated with this expectation is obtained from the law of Gaussian uncertainty propagation:

$$u_y = \left\{ \sum_{i=1}^{N} c_i^2 u_{xi}^2 + 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} c_i c_j u_{xixj} \right\}^{1/2} \qquad (6)$$

where $u_{xixj} = u_{xi} \cdot u_{xj} \cdot r_{ij}$ is the covariance of the quantities X_i and X_j, and r_{ij} is the respective correlation coefficient.

It should be emphasized that for practical uncertainty evaluation almost all measurement systems may be represented by linearized models. This might be explained by the fact that "good measurement systems" usually show (very) low responsivity to perturbing/influencing quantities (see case (a) in figure 2).

2.2. Non-linear models

In some cases, such as, for example, in processing large signals, certain elements of the measurement process cannot be satisfactory modeled by linear transmission elements. Usually, the response characteristic (transfer behavior) of non-linear elements of measurement systems is mathematically described by

- non-linear algebraic equations, e.g. polynomials,
- ordinary differential equations, or
- partial differential equations.

In practice, preferably polynomials are used. Appropriate polynomials that describe the response characteristic at the given or desired operating point may be obtained by expanding a Taylor series to higher-order terms [1]. Often, Volterra series are utilized that originate in distortion computation. They describe the output of a non-linear transmission element as the sum of the responses of a first order, second order, third order operator and so on. If applicable, each operator is described either in the time or frequency domain with its own transfer function called a Volterra kernel [8].

If periodic processes are to be described, such as, for example, roundness errors, the transfer function may be represented by Fourier-series or related expansions.

For uncertainty evaluation of measurements which are described by non-linear model equations, the ISO-GUM procedure [1] cannot be applied. In this case, the so-called propagation of distributions is offers an appropriate solution.

In accordance with the Bayesian concept of the GUM, the (unavoidably incomplete) knowledge about each contributing input quantity is to be expressed by means of a probability-density function (pdf) $g_{Xi}\left(\xi_i\right)$. The pdfs for the input quantities are obtained from the given information about them by utilizing the principle of maximum information entropy (pme) [9]. The pdf for the measurand $Y, g_Y\left(\eta\right)$, is given by the so-called Markov formula:

$$g_Y\left(\eta\right) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g_{X1},\dots,g_{XN}\left(\xi_1,\dots,\xi_N\right) \cdot \delta\left(\eta - f_M\left(\xi_1,\dots,\xi_N\right)\right) d\xi_1,\dots,d\xi_N \tag{7}$$

where η are the possible values of the measurand *Y*.

From the above pdf, the expectation $y = E\left[Y\right]$ and its associated uncertainty can be calculated as follows:

$$y = \int_{-\infty}^{+\infty} g_Y\left(\eta\right)\eta \cdot d\eta, \text{ and} \tag{8}$$

$$u_y = \left\{ \int_{-\infty}^{+\infty} g_Y\left(\eta\right)\left(y-\eta\right)^2 \cdot d\eta \right\}^{1/2} \tag{9}$$

Because equation (9) can analytically be computed in fairly simple cases only, modern uncertainty evaluation utilizes Monte-Carlo techniques as integration method [10]. Figure 3 illustrates this concept of pdf propagation.

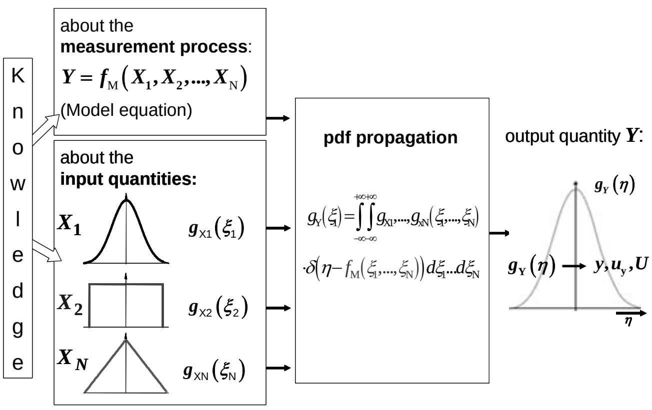


Fig. 3. - Illustration of the concept of pdf propagation [10]. Symbols see text.

From the above basic relationships on uncertainty evaluation, it may be concluded that independent of the method used – Gaussian uncertainty propagation or pdf propagation – an appropriate model equation is an indispensable prerequisite of modern uncertainty evaluation.

3. PRACTICAL MODELING PROCEDURE

3.1. Modeling steps

The practical modeling procedure is structured into five elementary steps:

- *1st step*: Describing the measurement.
- *2nd step*: Analyzing the measurement, decomposing it into its functional constituents and, in turn, establishing graphically the cause-and-effect relationship for the fictitious ideal (unperturbed) measurement in terms of standard modeling components.
- *3rd step*: Inserting all imperfections and effects that may perturb the fictitious ideal measurement, representing the cause-and-effect relationship graphically, and, in turn, mathematically for the real measurement. The insertion of imperfections, for example external influences, incomplete knowledge about parameters and instabilities, is to be carried out by utilizing correction factors and deviations from the fictitious ideal (nominal) parameters.
- *4th step*: Identifying and including correlation [11]. There are two possible ways to include correlation: *1st way*: If correlation is caused by conjoint functional dependencies on a third quantity, for example on temperature, these dependencies are to be accounted. The way to do this is to introduce these dependencies in the graphical and, in turn, mathematical cause-and-effect relationship and, therewith, the correlation will be dissolved. This first way is to be preferred. *2nd way*: Correlation is taken into account in accordance with the law of Gaussian uncertainty propagation (see equation (6)). This way, however, requires the knowledge of the estimated or experimentally determined value of the correlation coefficient [12].
- *5th step*: Inverting the mathematical cause-and-effect relationship to derive the relationship between the output quantity and the relevant input quantities, i.e. the model equation.

3.2. Standard modeling components

For the required graphical depiction of the cause-and-effect relationship, only three types of standard modeling components are employed: Parameter sources (SRC) to provide or reproduce a measurable quantity, transforming units (TRANS) to represent any kind of parameter processing and influencing and indicating units (IND) to indicate their input quantities [5] (see figures 4 to 7).

3.3. Example

The modeling procedure is explained with the simplified example of the calibration of a scale [12].

- *1st step*: Describing the measurement/calibration. A non-automatic scale is to be calibrated by means of a standard weight. This is carried out under prescribed conditions by direct measurement and comparison of the indication with the value of the standard given in the calibration certificate.

  Measurand: Instrumental error of the scale

  Causal quantity: Weighing value of the standard used

Measurement method: Direct measurement

- *2nd step*: Analyzing the measuring process. The standard may be considered as parameter source. Its imperfect "coupling" with the scale, e.g. caused by air buoyancy, magnetic susceptibility etc. might be described by a transforming unit. For a simplified treatment, the scale itself may be represented by an indicating unit. Figure 4 graphically represents the idealized (unperturbed) cause-and-effect relationship of the above described calibration.

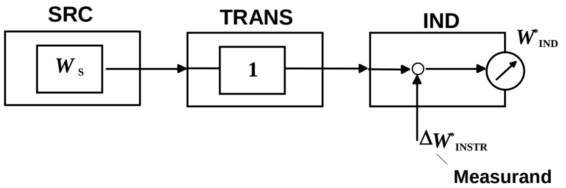


Fig. 4. - Cause-and-effect relationship for a fictitious ideal calibration of a scale. Symbols: W_S - weighing value reproduced by the standard; W^*_{IND} - indicated quantity; ΔW^*_{INSTR} - measurand (instrumental error of the scale to be calibrated).

- *3rd step*: Graphical depiction of the cause-and-effect relationship of the real measurement. By means of deviations and correction factors, the following influences and imperfections are to be introduced into the graphical cause-and-effect relationship of the described calibration: $\Delta W_S = W_{S0} - W_S$ is the error of the nominal value of the standard used; $k_B = \left(1 - \rho_a \cdot \rho_S^{-1}\right) / \left(1 - \rho_{1,2} \cdot \rho_{8000}^{-1}\right)$ - air buoyancy factor, ρ_a - air density, ρ_S - density of the standard, $\rho_{1,2} = 1{,}2$ kgm^{-3}, $\rho_{8000} = 8000$ kgm^{-3}, $\delta W_{CPL}(P)$ - deviation due to the influences of the (temperature-dependent) air convention and the magnetic field strength H ; $\delta W_M(t_a)$ - deviation of the scale due to the influence of the ambient temperature t_a. Figure 5(a) illustrates the real calibration of the scale and the appearing influences. Figure 5(b) shows the cause-and-effect relationship modeled for the real measurement. Expressed in mathematical terms, the cause-and-effect relationship of the real measurement reads:

$$\begin{aligned} W_{IND} &= \left(W_{S0} - \Delta W_S\right) k_B + \delta W_{CPL}(P) \\ &+ \delta W_M(t_a) + \Delta W_{INSTR} + \Delta W_{IND} \end{aligned} \tag{10}$$

- *4th step*: Identifying and including correlation. For the sake of simplification of this example, all involved quantities and observations are assumed to be independent of each other.

- *5th step*: Model equation. From the cause-and-effect relationship of the real measurement (see equation (10)), the following model equation is obtained:

$$\begin{aligned} \Delta W_{INSTR} &= W_{IND} - \left(W_{S0} - \Delta W_S\right) k_B \\ &- \delta W_{CPL}(P) - \delta W_M(t_a) + \Delta W_{IND} \end{aligned} \tag{11}$$

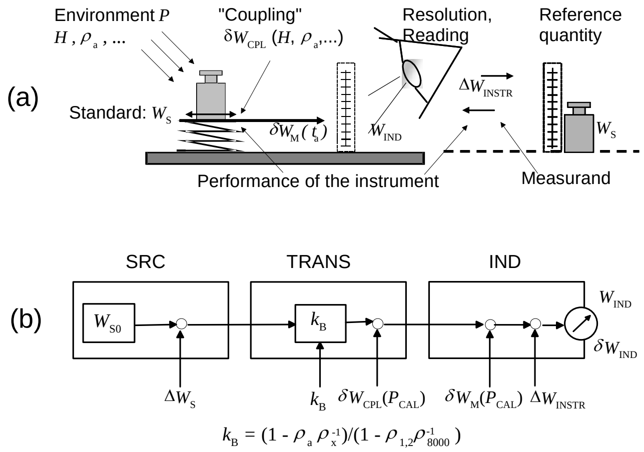


Fig. 5. - (a) Simplified example of a calibration of a scale. (b) Cause-and-effect relationship modeled for the real measurement. Symbols: W_S - weighing value provided by the standard; W_{S0} - nominal weight; ΔW_S , ΔW_{INSTR} - instrumental errors; ρ_a , ρ_x - densities of the air and the weight; δW_{CPL} - imperfect coupling effect of the quantity W_S with the instrument; δW_M - deviation representing the effect of calibration conditions P_{CAL} ; W_{IND} - indication; δW_{IND} - deviation due to the resolution; H - magnetic field strength [12].

After modeling the measurement, the most important step in uncertainty evaluation is to evaluate all involved input quantities that appear on the right-hand side of equation (11) by assigning appropriate probability density functions (pdfs) to them. Due to the almost linear model equation of the chosen example, the ISO-GUM method [1] may be used to determine the measurement uncertainty. As to the knowledge about the input quantities: W_S will be clearly indicated. The values of δW_{IND} can be derived from the instrument's resolution. Information about W_{S0} and ΔW_S should be given in the respective calibration certificate issued for the standard. The knowledge about $\delta W_{CPL}(P)$ and $\delta W_M(t_a)$ may be taken up from the manufacturer's manual or from the requirements set up in the European Standard EN 45501. The parameter k_B is to be estimated based on the knowledge about the ambient conditions and about the standard used.

4. MODEL STRUCTURES AND MEASUREMENT METHODS

Almost all measurement and calibrations can be reduced to only a few generic structures of cause-and-effect relationships. The structure and the chaining sequence of the modelling components in cause-and-effect relationships are determined by the method of measurement used [5]. Direct measurements result in an un-branched chain of the components utilized (see figure 5).

Other measuring methods are used to achieve higher accuracies and to ensure proper traceability of calibration results. These methods mostly result in branched cause-and-effect chains. Examples are given with the direct comparison of two indicating measuring instruments and the substitution method. Figure 6 and 7 show the generic structures of their cause-and-effect relationships. When deriving the mathematical cause-and-effect relationship from block diagrams having branched structures, for example the above methods, for each branch a separate partial equation is to be derived [5]. Thereby, influences and imperfections of the commonly used parts of the paths should be assumed to be correlated [11].

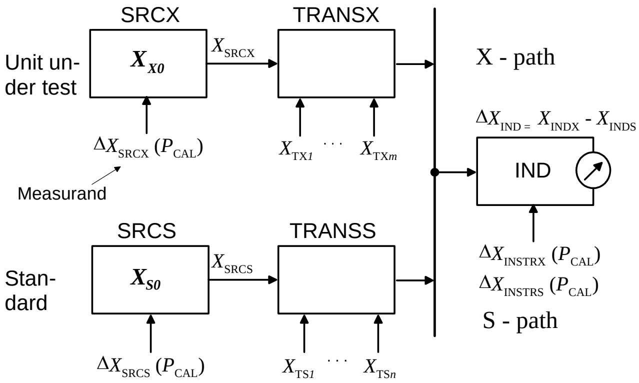


Fig. 6. Generic structure (conjoining chain) of a calibration by substitution. Symbols [5]: P_{CAL} -calibration conditions; X_{X0}, X_{S0} - nominal values for the material measures SRCX, SRCS; $\Delta X_{INSTR}(P)$ - instrumental error at the operating conditions P ; ΔX_{IND} - indicated quantity; ΔX_{SRCX} , ΔX_{SRCS} , ΔX_{INSTRX} , ΔX_{INSTRS} - instrumental errors; $X_{TX1},...,X_{TXm}$, $X_{TS1},...,X_{TSn}$ - input (influence) quantities.

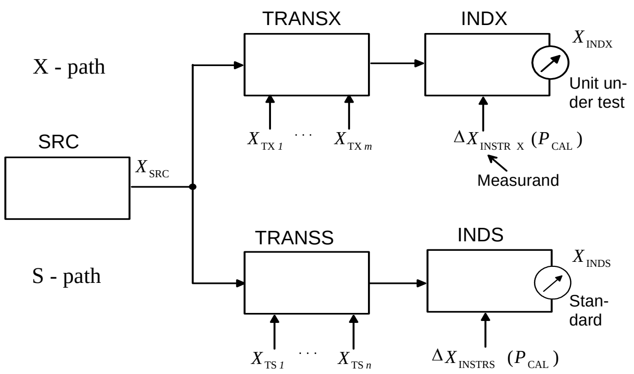


Fig. 7. Generic structure of the cause-and-effect relationship of a calibration by direct comparison of two indicating measuring instruments (forking chain) [5]. Symbols see figure 6.

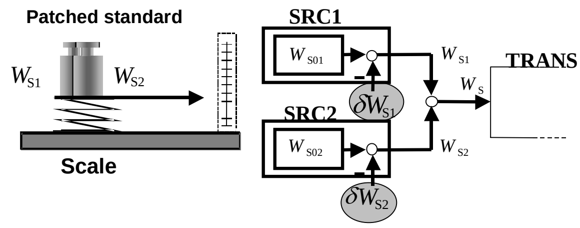
5. CORRELATED QUANTITIES IN MODELING

Correlation is present in many measurements and, dependent on the relationship of the correlated quantities, it enhances or decreases the combined uncertainty.

In practice, input quantities are often correlated because the same physical measurement standard, measuring instrument, reference datum, or even measurement method having a significant uncertainty is used in the estimation of their values [1].

When modeling measurements and calibrations in accordance with the above procedure (see section 3), it is an indispensable prerequisite that the input quantities assumed to be correlated really appear in the (graphical depiction of the) cause-and-effect relationship and in the model equation respectively.

If, for example, a standard weight W_S is established by two individual (patched) weight pieces, this is clearly to be represented in the graphical cause-and-effect relationship (see Figure 8). In case of this example, the (unknown) deviations of the patched standards used, ΔW_{S1} and ΔW_{S2}, are correlated quantities due to the presumed calibration of the standards within the "same experiment" and in the same laboratory respectively. It would not be a correct way to *a priori* combine the two standards or their deviations.



(a) (b)

Fig. 8. (a) Illustration of the application of "patched standards" in calibration. (b) Respective cause-and-effect relationship (cut-out). Symbols: W_{S01}, W_{S02} - nominal values of the partial standards used; ΔW_{S1}, ΔW_{S2} – instrumental errors of the standards used (correlated quantities because they have been determined within the same experiment or calibration)

The mathematically expressed cause-and-effect relationship for patched standards, derived for example from Figure 8(b), will always contain correlated quantities having identical signs. For the example depicted in Figure 8, it would be

$$W_S = W_{S01} + W_{S02} - \Delta W_{S1} - \Delta W_{S2} . \qquad (12)$$

Correlation of quantities that are linked additively or, always results in an enhanced combined uncertainty [11]. This combined uncertainty can easily be derived from the Gaussian law of uncertainty propagation [1] that, for the example of two correlated quantities X_1 and X_2, yields

$$\left[u_{x1}^2 + u_{x2}^2\right]^{1/2} \le u_{xTOTAL} \le u_{x1} + u_{x2} , \qquad (13)$$

where u_{x1} and u_{x2} are the individual uncertainty contributions, and u_{xTOTAL} is the total uncertainty contribution associated with the combined expectation value of the quantities X_1 and X_2.

In case of different signs of two correlated quantities X_1 and X_2 (see substitution example depicted in Figure 9;

correlated quantities: X_1 and X_2), or if correlated quantities are multiplicatively related correlation yields a decreased total uncertainty contribution:

$$0 \le u_{xTOTAL} \le \left[u_{x1}^2 + u_{x2}^2 \right]^{1/2} \qquad (14)$$

In case of the example depicted in Figure 9, the uncertainty would completely disappear in case that the instrumental error of the comparator is of systematic nature and absolutely stable. This, by the way, is exactly what substitution aims at.

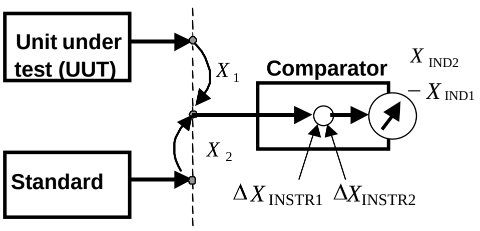


Fig. 9. Example: Graphical depiction of the cause-and-effect relationship (simplified) of a substitution measurement. Symbols: X_{S1}, X_{S2} - quantities provided by the material measures; ΔX_{INSTR1}, ΔX_{INSTR2} - (correlated) errors of the comparator used (separately) for the measurement of the standard and the unit under test; X_{IND1}, X_{IND2} - indicated quantities.

7. CONCLUSION

Although if seems not possible to develop a theory that allows for a stringent construction of a model, it is, nevertheless, possible to achieve systematic modeling based on the presented concept. Systematic modeling may be seen as an important improvement of uncertainty evaluation. The particular concept is applicable to most areas of uncertainty evaluation of measurements performed.

REFERENCES

[1] International Organization for Standardization (ISO), “Guide to the Expression of Uncertainty in Measurement” (GUM), Geneva, first edition 1993 corrected and reprinted 1995.

[2] H. Bachmair, “A simplified method to calculate the uncertainty of measurement for electrical calibration and test equipment”, Proceedings of the 9th International Metrology Congress Bordeaux 18 - 21 October 1999, Mouvement Francais pour la Qualite, Nanterre Cedex, (ISBN 2-909430-88x).

[3] W. Kessel, „Messunsicherheit - einige Begriffe und Folgerungen fuer die messtechnische Praxis“, PTB-Mitteilungen, vol. 111, pp. 226-244, Wirtschaftsverlag NW, Bremerhaven, 2001 (ISSN 030-834X).

[4] D. Kind, „Die Kettenschaltung als Modell zur Messunsicherheitsberechnung“, PTB Mitteilungen, vol. 111, pp. 338-341, Wirtschaftsverlag NW, Bremerhaven, 2001 (ISSN 030-834X).

[5] K.-D. Sommer, M. Kochsiek, B.R.L. Siebert, A. Weckenmann, “A Generalized Procedure for Modelling of Measurement for Evaluating the Measurement Uncertainty”, Proceedings of the 17th IMEKO World Congress 2003, Dubrovnik (ISBN 953-7124-00-2).

[6] U. Kiencke, H. Jaekel, “Signale und Systeme”, 3rd edition, Oldenbourg, Muenchen, 2005.

[7] O. Mildenberger, M. Meyer, “Grundlagen der Informationstechnik - Signale, Systeme und Filter”, Vieweg, Braunschweig, 2002.

[8] V. Kafka, “Rekursive Strukturen auf Volterra-Basis zur aufwandsarmen Darstellung und Entzerrung von nichtlinearen Systemen”, dissertation, Universitaet der Bundeswehr Muenchen, 2002.

[9] W. Weise, W. Woeger, “Messunsicherheit und Messdatenauswertung”, WILEY-VCH, Weinheim, 1999.

[10] B.R.L. Siebert, K.-D. Sommer, “New Developments of the GUM and Monte Carlo Techniques”, Technisches Messen, vol. 71, no. 2, pp. 67-80, 2004.

[11] K.-D. Sommer, M. Kochsiek, B.R.L. Siebert, “A Consideration of Correlation in Modelling and Uncertainty Evaluation of Measurement”, Conference Proceedings NCSL International Boulder, NCSL International Workshop and Symposium, Salt Lake City, 2004.

[12] K.-D. Sommer, M. Kochsiek, B.R.L. Siebert, “Modelling of Weighing Procedures for Uncertainty Evaluation”, Conference Proceedings South Yorkshire Trading Standards Unit, Chapeltown, South Yorkshire International Conference Sheffield, 2003, Barnsley.