XVIII IMEKO WORLD CONGRESS
Metrology for a Sustainable Development
September, 17 – 22, 2006, Rio de Janeiro, Brazil

ON THE USE OF THE UNSCENTED TRANSFORM FOR UNCERTAINTY EVALUATION IN DSP-BASED MEASUREMENTS

Leopoldo Angrisani[1], *Aldo Baccigalupi*[1], *Massimo D'Apuzzo*[2], *Rosario Schiano Lo Moriello*[1]

[1] Dipartimento di Informatica e Sistemistica, [2] Dipartimento di Ingegneria Elettrica
Università degli Studi di Napoli Federico II, Via Claudio 21, 80125 Napoli, Italia.
Ph: +39-081-7683170, Fax: +39-081-2396897
E-mail: {angrisan, baccigal, dapuzzo, rschiano}@unina.it

Abstract: Estimation of the output expectation and standard uncertainty in indirect measurements based on digital signal-processing (DSP) algorithms is dealt with. DSP measurement algorithms can, in fact, be considered as complex models, which operate on sampled data acting as direct measurement results. The related uncertainty sources can mainly be associated with the analog conditioning and analog-to-digital conversion the input signal undergoes. Due to the inherent non-analytical formulation of DSP measurement algorithms, the use of IEC-ISO recommendation for standard uncertainty estimation is usually unfeasible. The paper aims at showing the efficacy and reliability of an original approach, mandated to uncertainty estimation and based on the unscented transform, when applied to DSP measurement algorithms.
A number of numerical tests are conducted on simulated measurement data. Remarkable concurrence between obtained estimates and those granted by Monte Carlo simulations confirms the efficacy of the proposed approach.

Keywords: Uncertainty estimation, DSP-based measurements, Analog-to-digital conversion.

1. INTRODUCTION

The availability of more and more performing data acquisition systems and digital signal processors have allowed a massive use of digital signal-processing algorithms in a wide range of measurement applications. Any measurement performed through a digital signal-processing (DSP) algorithm can be considered as indirect; the final result depends on the value of the acquired samples of one or more input signals. The samples are obtained as a result of direct measurements, and the DSP measurement algorithm stands for the relationship between the output (final result) and input (input signal samples) quantities, modeled as random variates.

According to the recommendations of ISO-IEC Guide to the Expression of Uncertainty in Measurement (GUM) [1], [2], an uncertainty value must be associated to the estimate of the expectation of each output quantity. As shown by remarkable examples presented in [3] and [4], the GUM provides a suitable law for combining the uncertainties associated with the estimates of the expectation of input signal samples. In particular, the best estimates of expectation and standard deviation of the probability density functions (pdf's) characterizing the random variates modeling the same input quantities are first gained [1],[2]. This information is then "propagated" through a first-order Taylor series approximation of the measurement model (involving the calculation of the first order derivatives of the measurement model) to provide an estimate of output expectation and standard uncertainty.

In the presence of complex DSP measurement algorithms, the application of the IEC-ISO Guide may, however, become quite troublesome. To overcome this problem, some approaches based on Monte Carlo simulations [5] and random-fuzzy variables [6] have been proposed. Even though capable of granting reliable estimates of the expectation and standard uncertainty of output quantities, they suffer from long implementation time and/or heavy computational burden [6].

The authors have already proposed in [7] and [8] an original approach, based on the unscented transform (UT), for the estimation of output expectation and standard uncertainty in indirect measurements. Input quantities, in particular, can be modeled both as uncorrelated and correlated random variates. Thanks to suitable properties of the UT, the approach requires a reduced computational burden to assure as reliable estimates as those provided by a huge number of Monte Carlo simulations.

The paper aims at showing the efficacy and reliability of the proposed approach also when applied to measurements based on digital signal processing algorithms, whatever their complexity. In particular, the authors focus their attention on the influence that the analog-to-digital converter (ADC) has on the estimate both of expectation and standard uncertainty of each input sample. Starting from well-known expressions that relate ADC codes to the corresponding analog values [9], the proposed approach is capable of provided as good estimates of the output expectation and standard uncertainty as those that would be granted by high order Taylor series approximations.

In the following, after a brief outline of the theoretical background of the proposed approach, the fundamental steps of its application to DSP measurement algorithms are described in detail. Results obtained in a number of tests involving different DSP measurement algorithms are presented, and compared to those provided by Monte Carlo simulations, taken as reference.

2. THE PROPOSED APPROACH

Let us consider a vector $\underline{\boldsymbol{X}} = \left[X_1, X_2, \ldots, X_N\right]$ of uncorrelated random variates, each of which is characterized by its own pdf and models an input quantity of an indirect measurement $Y = f\left(X_1, X_2, \ldots, X_N\right)$. For each X_i (i=1,..,N), its expectation and a suitable collection of central moments have firstly to be estimated. This task can easily be fulfilled either through repeated measurements or already available information and/or user knowledge and experience.

According to UT theory, the matrix $\boldsymbol{\chi}$ the columns of which are the so-called sigma points, has then to be arranged:

$$\boldsymbol{\chi} = \left[\mathbf{X}+\boldsymbol{\Sigma}_1 \quad \mathbf{X}+\boldsymbol{\Sigma}_2 \quad .. \quad \mathbf{X}+\boldsymbol{\Sigma}_G \quad \overline{\mathbf{x}}^T\right] \quad . \quad (1)$$

$\boldsymbol{\chi}$ has N rows and $G{\cdot}N{+}1$ columns, where G is the number of considered central moments. Vector $\overline{\mathbf{x}}$, the last column of $\boldsymbol{\chi}$, contains the best estimates (x_i, i=1,..,N) of the expectation of the variates X_i (i=1,..,N). Moreover, $\mathbf{X}$ and $\boldsymbol{\Sigma}_j$, j=1,..,G, are N-dimensional, square matrices given respectively by:

$$\mathbf{X}=\begin{pmatrix} x_1 & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ x_N & \cdots & x_N \end{pmatrix} \text{ and } \boldsymbol{\Sigma}_j = \begin{pmatrix} s_{j1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & s_{jN} \end{pmatrix} \quad (2)$$

obtained by solving the following nonlinear equation system:

$$\begin{cases} 1 = W_0 + N(W_1 + W_2 + .. + W_G) \\ \mathbf{0} = W_1\boldsymbol{\Sigma}_1 + W_2\boldsymbol{\Sigma}_2 + .. + W_G\boldsymbol{\Sigma}_G \\ \boldsymbol{\mu}^2 = W_1\boldsymbol{\Sigma}_1^2 + W_2\boldsymbol{\Sigma}_2^2 + .. + W_G\boldsymbol{\Sigma}_G^2 \\ \ldots \\ \boldsymbol{\mu}^G = W_1\boldsymbol{\Sigma}_1^G + W_2\boldsymbol{\Sigma}_2^G + .. + W_G\boldsymbol{\Sigma}_G^G \end{cases} \quad . \quad (3)$$

$\boldsymbol{\mu}^k$ are NxN diagonal matrices, the generic element of which, μ_{ii}^k, is equal to the k^{th} central moment of the i^{th} input random variate. The value of each weight W_i can, in principle, be fixed arbitrarily; in practice, the choice should be helpful in solving the equation system (6) easily. Statistics of the resulting sigma points are inherently equal to those characterizing the vector $\boldsymbol{X}$. The function f is then applied to each sigma point: $\psi_j = f\left(\boldsymbol{\chi}\big|_j\right)$, $j = 1,..,GN+1$. $\boldsymbol{\chi}\big|_j$ stands for the j^{th} column of the matrix $\boldsymbol{\chi}$. All obtained values, ψ_j, are processed to estimate the output quantity expectation and standard uncertainty according to the following expressions:

$$\overline{y} = \sum_{j=1}^{G}\sum_{i=1}^{N} W_j\psi_{ji} + W_0\psi_{GN+1} \quad ; \quad (4)$$

$$\mu_{\overline{y}} = \sqrt{\sum_{j=1}^{G}\sum_{j=1}^{N} W_j\left(\psi_{ji} - \overline{y}\right)^2 + W_0\left(\psi_{GN+1} - \overline{y}\right)^2} \quad . \quad (5)$$

In the presence of input quantities modeled by random variates characterized by symmetric pdf's, the solution of equation system (3) can be optimized through the following choices:

$$\underline{\underline{\Sigma}}_{2k} = -\underline{\underline{\Sigma}}_{2k-1},\ W_{2k} = -W_{2k-1},\ k=1,..,\frac{G}{2} \quad . \quad (6)$$

This way, only $\frac{G}{2}$ equations along with related weights W_i and unknown matrices $\underline{\underline{\Sigma}}_i$ are necessary to assure that (i) even central moments, up to the G^{th} order, of resulting sigma points are equal to the corresponding ones of input variates and (ii) odd central moments are null according to the assumption of input random variates characterized by symmetric pdf's. Thanks to a proper selection of the values of W_i (W_1=W_3 and W_2=W_4 = -W_1), the authors are able to solve the equation system (3) in explicit form, for central moments up to the 8$^{\text{th}}$ order [7].

It is worth highlighting that the higher the number of considered input central moments, the more accurate the estimates of the output expectation and standard uncertainty [8]. Moreover, results similar to those granted by higher-order Taylor series approximations can be assured with no need of any derivative of f. This is a very attractive feature whenever the analytical form of f is not available, as in the presence of complex digital signal-processing algorithms.

3. APPLICATION TO DSP-BASED MEASUREMENTS

From a metrological point of view, any measurement based on a DSP algorithm can be considered indirect. The number of input quantities X_i, i=1,..,N, could however be high, thus making unacceptable the dimension of the matrix $\boldsymbol{\chi}$ containing the sigma points to be transformed. To reduce the number of sigma points needed, it has to be noted that each input sample X_i can be gained from the corresponding code k_i provided by the analog-to-digital converter (ADC), through a straightforward expression [9]:

$$X_i = \frac{k_i + Q_i}{T_i} + O_i \quad . \quad (7)$$

Q_i, T_i, and O_i are random variates modeling respectively the ADC quantization, gain and offset (referred to as ADC parameters); the variates depend only on the considered code k_i. The measurement model can thus be written as

$$Y = f\left(X_1,..,X_N\right) = g\left(Q_1,T_1,O_1,..,Q_M,T_M,O_M\right) \quad (8)$$

where M stands for the number of actually stimulated ADC codes. The proposed approach can successfully be applied to the new function $g\left(\cdot\right)$. In the worst case of an input signal capable of stimulating all ADC codes, the number of sigma points does not exceed 3GH+1, where H, equal to 2^n (n is the ADC resolution), is the whole number of ADC codes. It is worth noting that we focus only on the effect of the considered ADC parameters on the estimates of output expectation and standard uncertainty; no other effect due has been taken into account.

From an operative point of view, the application of the proposed approach enlists the following steps (Fig.1).

1. Digitized samples are first analyzed to determine the M ADC codes really involved.
2. For each code k_j (j = 1,..,M) singled out at the previous step, the sigma points associated with the random variates modeling the ADC parameters have to be evaluated. To this aim, an estimate of the needed central moments can easily be achieved either from technical ADC specifications or values that well-known statistical estimators exhibit when applied to the results obtained through independent, repeated measurements. It is worth stressing that, due to their inherently deterministic nature, the sigma points could be evaluated only once.
3. Voltage values corresponding to the digitized samples are attained by applying the equation (7) to the available expectation estimates of ADC parameters for all M codes, but the current code k_j. The values associated with k_j are, in turn, gained from the application of the same equation (7) to the $3G$ sigma points obtained in the previous step.
4. The DSP algorithm is applied to the attained voltage values, and the result, i.e. the transformed sigma points, is retained for the final estimation of output expectation and standard uncertainty to be achieved.
5. Steps 2, 3 and 4 are repeated until all the M different codes are considered.
6. Output expectation and standard uncertainty estimates are gained from the application of the equations (5) and (6) to the $3GM$+1 transformed sigma points obtained.

It is worth noting that, in the presence of DSP algorithms whose outputs are vectors or matrices, the required memory resources could be unacceptable; to estimate the standard uncertainty through the equation (5), all partial results ψ_j have, in fact, to be retained. To overcome this problem, the authors suggest an implementation trick very similar to the variance decomposition. In particular, the equation (5) can be written as:

$$\mu_{\bar{y}}^2 = \sum_{j=1}^{G}\sum_{j=1}^{N} W_j\left(\psi_{ji}^2 - 2\psi_{ji}\bar{y} + \bar{y}^2\right) + W_0\left(\psi_{GN+1}^2 - 2\psi_{GN+1}\bar{y} + \bar{y}^2\right) = \sum_{j=1}^{G}\sum_{j=1}^{N} W_j\psi_{ji}^2 + W_0\psi_{GN+1}^2 - 2\bar{y}\left(\sum_{j=1}^{G}\sum_{j=1}^{N} W_j\psi_{ji} + W_0\psi_{GN+1}\right) + \left(\sum_{j=1}^{G}\sum_{j=1}^{N} W_j + W_0\right)\bar{y}^2 \quad (9)$$

By substituting the expressions (3) and (4) in equation (7), the desired estimate of the standard uncertainty can be written as:

$$\mu_{\bar{y}} = \sqrt{\sum_{j=1}^{G}\sum_{j=1}^{N} W_j\psi_{ji}^2 + W_0\psi_{GN+1}^2 - \bar{y}^2} \quad . \quad (10)$$

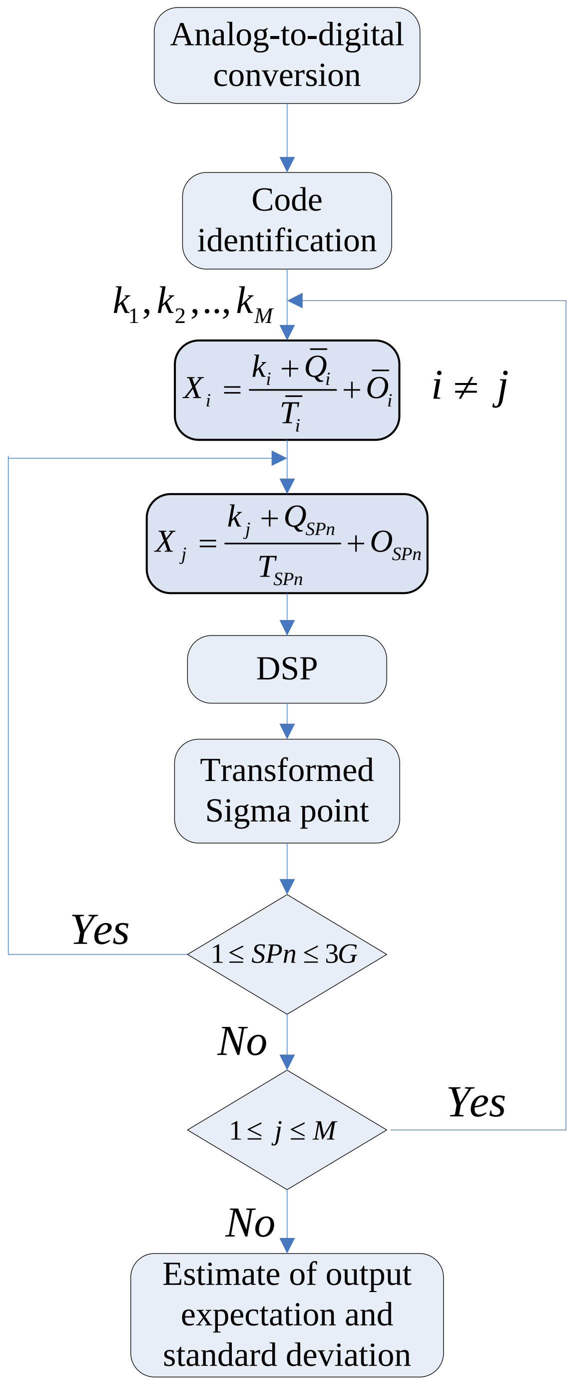

Fig.1 Operating steps of the proposed approach.

This way, it is necessary to retain only the cumulative weighted sum both of simple and squared transformed sigma points.

4. TESTS ON SIMULATED MEASUREMENT DATA

A number of tests have been conducted on simulated measurement data. Three different DSP algorithms have been considered, characterized respectively by scalar, vector, and matrix output. For each DSP algorithm, estimates of the output expectation, y, and standard uncertainty, μ, have been gained both through the proposed approach and Monte Carlo simulations.

Tab.I Results obtained in tests on the considered DSP-based algorithm. Δy and $\Delta\mu$ are expressed in percentage relative terms.

ER / N	32			64			128			256		
	M	Δy (%)	$\Delta\mu$ (%)	M	Δy (%)	$\Delta\mu$ (%)	M	Δy (%)	$\Delta\mu$ (%)	M	Δy (%)	$\Delta\mu$ (%)
256	32	0.0015	0.071	64	$-1.0\cdot10^{-4}$	-0.0087	102	$1.3\cdot10^{-4}$	0.13	116	$-2.6\cdot10^{-4}$	-0.045
512	32	$-2.3\cdot10^{-4}$	0.411	64	$2.7\cdot10^{-4}$	0.14	128	0.0013	0.027	204	$2.4\cdot10^{-4}$	-0.42
1024	32	$-4.0\cdot10^{-4}$	-0.78	64	-0.0025	0.0028	128	$-2.2\cdot10^{-4}$	0.031	256	$-2.4\cdot10^{-4}$	-0.16
2048	32	$1.1\cdot10^{-4}$	-0.41	64	$8.3\cdot10^{-4}$	-0.50	128	$-2.8\cdot10^{-4}$	-0.23	256	$-1.6\cdot10^{-4}$	0.017
4096	32	-0.0035	-0.45	64	0.0012	-0.12	128	$4.3\cdot10^{-4}$	-0.22	256	$-1.0\cdot10^{-4}$	0.13

For the sake of simplicity, the random variates modeling Q, T, and O have been supposed uniformly distributed; no additional computational burden is required if this assumption is not met. In particular, the quantization Q and offset O have been modeled as zero-mean, uniformly distributed random variates, characterized by the same width equal to $\frac{LSB}{2}$, while the gain T has been modeled as a uniformly distributed random variate with mean and width equal respectively to 1 and 0.01.

4.1. DSP algorithm with scalar output

A very common DSP algorithm has been considered. It is mandated to the evaluation of the root mean square value of an acquired waveform, according to the following relation:

$$Y = \sqrt{\frac{1}{N}\sum_{i=1}^{N} X_i} \quad . \tag{11}$$

Several tests have been conducted on simulated sinusoidal signals, the samples of which have been obtained in the assumption of an 8-bit, bipolar ideal ADC. All signals have been characterized by a different value both of the effective range ER (i.e. twice the maximum code acquired in the record) and record length N. Moreover, their normalized frequencies have been fixed equal to $\frac{1}{N}$ in order to operate in the hypothesis of coherent sampling. Achieved performance is given as differences Δy and $\Delta\mu$, both expressed in relative percentage terms, between the obtained estimates and those granted by K=10^5 Monte Carlo simulations.

Obtained results are given in Tab.I Remarkable concurrence can be noticed; differences never greater then 0.4% have, in fact, been experienced. It is worth noting that such a high value of K is necessary in order to assure reliable output estimates that can, thus, be taken as reference.

The huge number of Monte Carlo simulations is mainly due to their inherent random approach adopted for the estimation of output distribution. The variates modeling input quantities are, in fact, randomly sampled in their domain; this way, the higher the number of input samples, the more accurate the estimate both of expectation and standard uncertainty. For the sake of clarity, Fig.2 and Fig.3 give the outcomes of 100 different runs of Monte Carlo simulations (blue dots) for different values of K, i.e. 10, 30, 100, 300, 1000, 2544 (equal to $3GM+1$ with G=8 and M=106), 3000, and 10000. Fig.2 and Fig.3 highlight that, for low values of K, expectation and standard uncertainty estimates provided by Monte Carlo simulations are very poor and scattered about the correspondingly mean values, which are very close to those granted by the proposed approach (red stars). As expected, the reliability of the estimates improves upon the increasing of the number of Monte Carlo simulations, to the detriment of computational burden (Fig.4). On the contrary, the proposed approach is capable of granting as good estimates as those of 10^5 Monte Carlo simulations with a much reduced computational time (Fig.5).

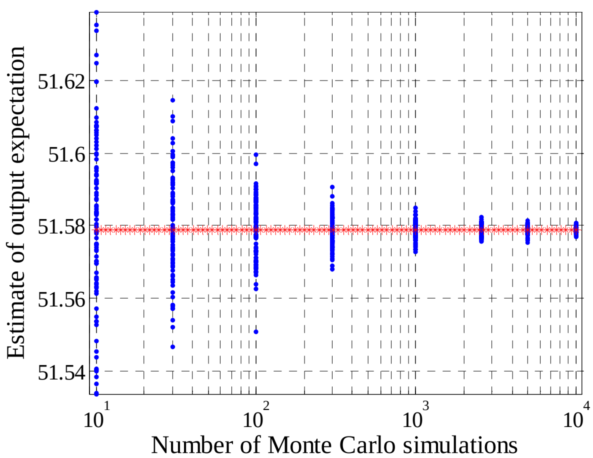

Fig.2 Outcomes provided by proposed approach (red star) and Monte Carlo simulations (blue dots) in 100 different runs, each of which is characterized by a different number of simulations.

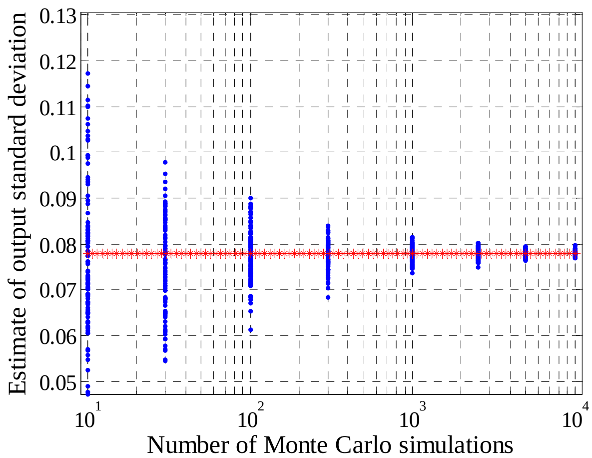

Fig.3 Outcomes provided by proposed approach (red star) and Monte Carlo simulations (blue dots) in 100 different runs, each of which is characterized by a different number of simulations.

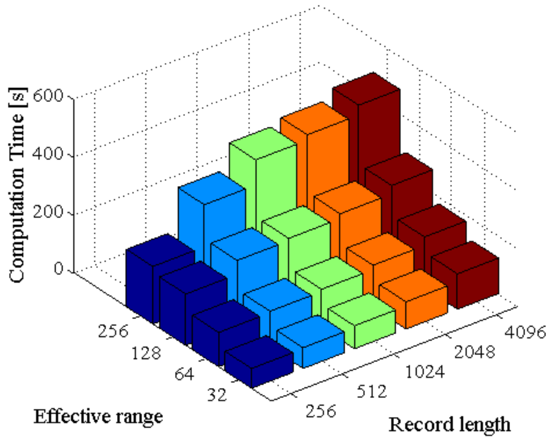

Fig.4 Computational time of 10^5 Monte Carlo simulations versus the effective range *ER* and record length *N*.

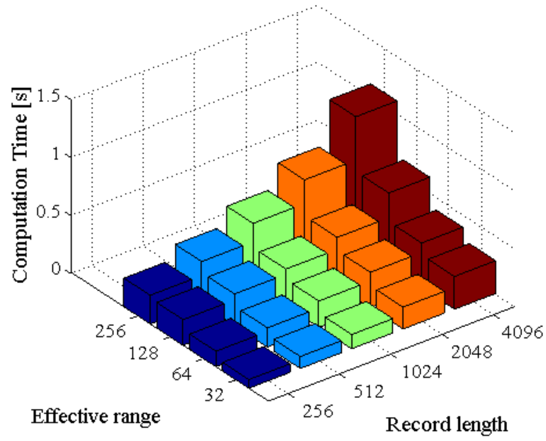

Fig.5 Computational time of the proposed approach versus the effective range *ER* and record length *N*.

4.2. DSP algorithm with vector output

The application of the proposed approach to a DFT (discrete Fourier transform) algorithm is presented in the following [10]. In particular, the evaluation of the magnitude spectrum versus the normalized frequency according to

$$\begin{aligned} &Y(0) = \left| \frac{1}{N} \sum_{n=0}^{N-1} X_n \right| \\ &Y\left(\frac{i}{N}\right) = \left| \frac{2}{N} \sum_{n=0}^{N-1} X_n e^{-j2\pi \frac{i}{N} n} \right| \quad i = 1,..N-1 \end{aligned} \quad . \qquad (12)$$

has been pursued. Several tests have been conducted on simulated sinusoidal signals, characterized by different value both of the effective range *ER* and record length *N*. Achieved performance has been expressed both as RMS and maximum value of the difference vectors, Δy and $\Delta\mu$, between the obtained estimates and those granted by 10^5 Monte Carlo simulations. RMS values has, in particular, been evaluated according to the relations

$$\begin{aligned} &RMS\Delta y = \sqrt{\frac{1}{N} \sum_{i=1}^{n} \Delta y_i} \\ &RMS\Delta \mu = \sqrt{\frac{1}{N} \sum_{i=1}^{n} \Delta \mu_i} \end{aligned} \quad . \qquad (13)$$

A valuable accord between the estimates provided by the proposed approach and those granted by Monte Carlo simulations has been encountered also in these tests. RMS values normalized to the signal amplitude never greater than 0.002 and a maximum difference, in percentage relative terms, always lower than 0.2% have been experienced. As an example, Fig.10 shows the results provided by the proposed approach when applied to a sinusoidal signal whose effective range and normalized frequency have been respectively equal to 256 and $1/\sqrt{200}$. Obtained estimates are superimposed to those granted by Monte Carlo simulations. The concurrence is still more evident if the difference vectors Δy and $\Delta\mu$ are considered (Fig.9).

4.3. DSP algorithm with matrix output

As a final example, the results obtained when the proposed approach is applied to a standard short time

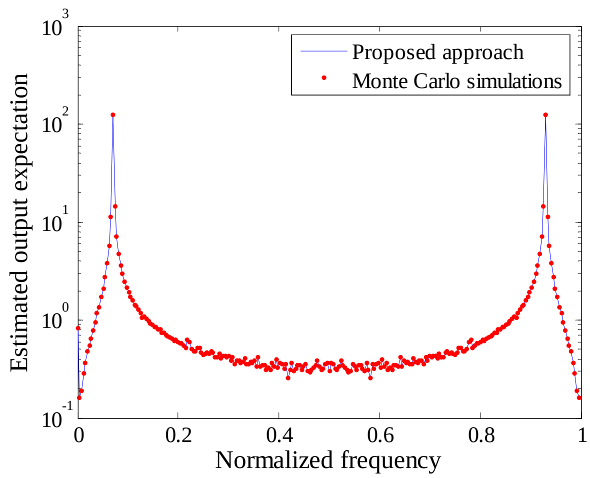

Fig.6. Comparison of expectation estimates provided by the proposed approach to those granted by 10^5 Monte Carlo simulations, for each spectral line.

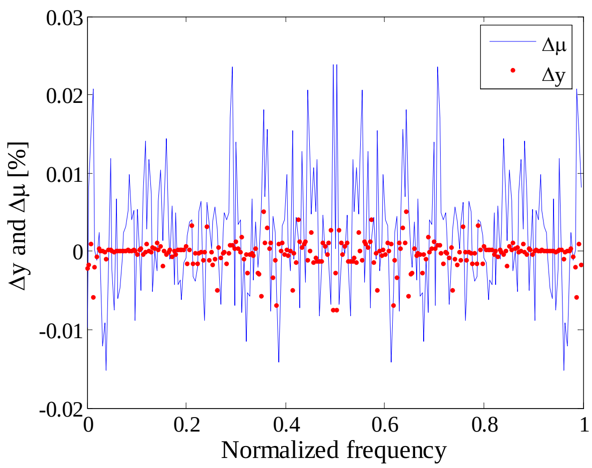

Fig.7. Δy and $\Delta\mu$, expressed in percentage relative terms, versus normalized frequency.

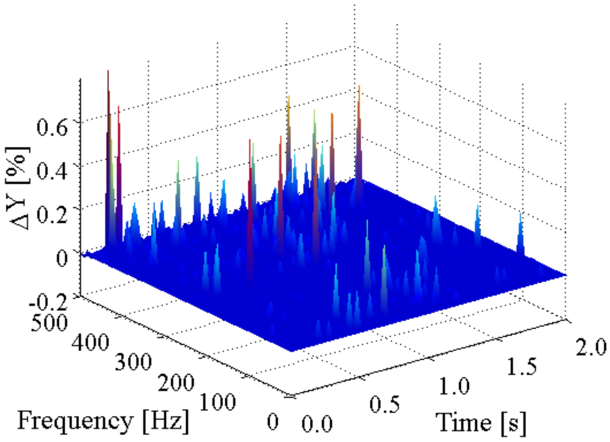

Fig.8. Differences, for each point of time-frequency plane, between the expectation estimates of the STFT coefficients provided by the proposed approach and those granted by 10^4 Monte Carlo simulations.

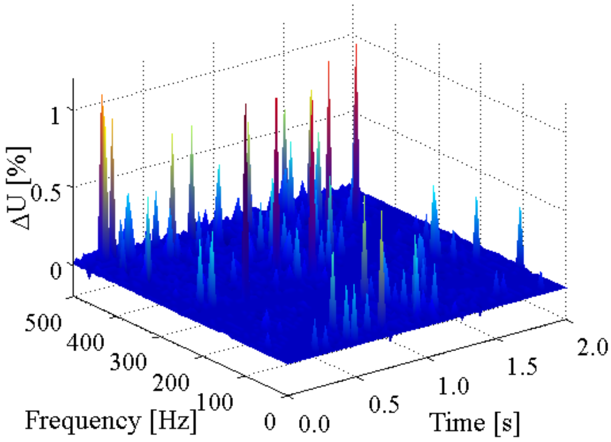

Fig.9. Differences, for each point of time-frequency plane, between standard uncertainty estimates of the STFT coefficient provided by the proposed approach and those granted by 10^4 Monte Carlo simulations.

Fourier transform (STFT) [10] are presented. The analyzed signal is the sum of two interfering chirps, both consisting of 4000 samples. In particular, Fig.8 and Fig.9 show the differences, for each point of time-frequency plane, between the expectation and standard uncertainty estimates of STFT coefficients, taken in modulus, provided by the proposed approach and those granted by 10^5 Monte Carlo simulations. Remarkable concurrence can be noticed; percentage relative differences always lower than 1% has, in fact, been experienced. As regards the computational time, the proposed approach provides the desired estimates in 12 s; on the contrary, Monte Carlo simulation takes over 40 minutes.

4. CONCLUSION

An original approach for the uncertainty evaluation in indirect measurements based on digital signal-processing algorithms has been presented. The approach exploits nice properties of the unscented transform in order to provide reliable estimates of the output quantity. In particular, the authors have focused their attention on the influence of the analog-to-digital conversion section on the final output.

Several tests conducted in the presence of different DSP measurements algorithms have been conducted in order to assess the reliability of the proposed approach. For each test, the results provided by the proposed approach have been compared with those granted by a suitable number of Monte Carlo simulations. Remarkable concurrence has been noticed; differences between obtained estimates and those granted by Monte Carlo simulations have, in fact, been always lower than 0.7%.

Future activity is mainly oriented to the statistical characterization of actual analog-to-digital converters in order to experimentally achieve the relation between any code and the corresponding voltage values.

REFERENCES

[1] BIPM, IEC, IFCC, ISO, IUPAC, and OIML, “Guide to the expression of uncertainty in measurement,” 1993.

[2] BIPM, IEC, IFCC, ISO, IUPAC, and OIML, “Guide to the expression of uncertainty in measurement, supplement 1, numerical methods for the propagation of distributions,” 2004.

[3] G. Betta, C. Liguori, and A. Pietrosanto, “Propagation of Uncertainty in a Discrete Fourier Transform Algorithm,” Measurement, Vol. 27, pp. 231-239, 2000.

[4] L. Peretto, R. Sasdelli, and R. Tinarelli, “On Uncertainty in Wavelet-Based Signal Analysis,” IEEE Trans. on Instrum. and Meas., Vol. 54, No. 4, August 2005.

[5] N. Locci, C. Muscas, L. Peretto, and R. Sasselli, “A Numerical Approach to the Evaluation of Uncertainty in Nonconventional Measurement on Power System,” IEEE Trans. on Instrum. and Meas., Vol. 51, No. 4, pp.734-739, August 2002.

[6] A. Ferrero, S. Salicone, “A Comparative Analysis of the Statistical and Random-fuzzy Approaches in the Expression of the Uncertainty in Measurement,” IEEE Trans. on Instrum. and Meas., Vol. 54, No. 4, pp.1475-1481, August 2005.

[7] L. Angrisani, M. D’Apuzzo, and R. Schiano Lo Moriello, “The Unscented Transform: a Powerful Tool for Measurement Uncertainty Evaluation,” IEEE Trans. on Instrum. and Meas., Vol. 55, No. 3, June 2006.

[8] L. Angrisani, M. D’Apuzzo, and R. Schiano Lo Moriello, “New Proposal for Uncertainty Estimation in Indirect Measurements with Correlated Input Quantities,” IEEE Trans. on Instrum. and Meas., Vol. 55, No. 4, August 2006.

[9] A. Ferrero, R. Gamba, S. Salicone, “A Method Based on Random-Fuzzy Variables for Online Estimation of the Measurement Uncertainty of DSP-Based Instruments,”, IEEE Trans. on Instrum. and Meas., Vol. 53, No. 5, pp.1362-1369, October 2004.

[10] V.K. Madisetti, D.B. Williams, “Digital Signal Processing Handook,” CRC Press, 1999.