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ACCURACY OF STATISTIC AND SPECTRAL MEASUREMENTS

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Abstract − The paper is concerned with attempts to investigate the influence of measurement results errors, the algorithms for signal values recovering between the measurement actions, time discrete period, the samples number, and the interval of signal observation as well as other factors on the procedures of statistic treatment of measurements results. The application of the obtained procedures for measurement precision assessment to knowledge base creation provides a possibility to increase the intelligent measurement efficiency.

Keywords: statistic measurements, algorithms, error, combined approach.

1. INTRODUCTION

The assessment of algorithms precision is an essential element in the knowledge base for intelligent measurements. Efficiency of such measurements depends on adequacy and accuracy of these assessments [1].

As a rule, each factor influencing the error of statistic measurements is assessed by its elementary error; then these errors are summed up. Analysis of this procedure proves that the accuracy of such assessments is not high, as each elementary error is calculated on the basis of its mathematical model, and their summation practically takes no account of their being correlated [2, 3].

Methods of simulative modelling and experimental determination of error characteristics are devoid of the disadvantage mentioned above. However, they require much time and availability of expensive reference means of measurement. Besides, experimental methods are known to have limited functional potentials. Hence, the combined use of all of these methods appears to have considerable promise for intelligent measurements [1].

2. ESSENCE OF THE COMBINED APPROACH TO ERROR DEFINITION

It is quite possible to enhance the capabilities of calculation techniques by taking into account the main factors influencing the statistic measurements accuracy: readings errors, algorithms of signal restoration between readings, spacing of time sampling, sample size, realization length. The idea of the combined approach implies that the errors of the statistic measurements are considered as an indivisible unit, which is transformed with the change of measurement modes, environment factors, etc. [4].

3. APPLICATION OF THE ERGODIC PROPERTY FOR THE MEASUREMENT OF PROBABILITIES DENSITES OF RANDOM PROCESSES

For ergodic random processes, the assessments of one-dimentional $\langle\varpi_1[X]\rangle$ and of two-dimentional $\langle\varpi_2[X',X'']\rangle$ probabilities densities are equal to [5]:

$$\langle \varpi_1[X]\rangle = \frac{1}{\mathrm{T}}\int_0^T \varpi_1\left[X\,|\langle x(t)\rangle\right]dt \;; \tag{1}$$

$$\langle \varpi_2[X',X'']\rangle = \frac{1}{T-|t''-t'|}\int_0^{T-|t''-t'|} \varpi_2\left[X',X''|\langle x(t)\rangle\right]dt \;, \tag{2}$$

where $\langle x(t)\rangle$ is assessment of random process realization measured with analogue method; T is its duration.

In digital measurements with the spacing of time sampling $T_0=t_{i+1}-t_i$ where i is the number of the reading, and extrapolation of the signal $x(t)$ by one previous reading x_i, expressions (1) and (2) take the form [6] :

$$\langle \varpi_1[X]\rangle = \frac{1}{nT_0}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} \varpi_1\left[X\,|x_i\right]dt \;; \tag{3}$$

$$\langle \varpi_2[X',X'']\rangle = \frac{1}{(n-\mu)T_0}\sum_{i=1}^{n-\mu}\int_{t_i}^{t_{i+1}} \varpi_2[X',X''|x_i]dt, \tag{4}$$

where $\varpi_1[X|x_i]$ and $\varpi_2[X',X''|x_i]$ are one-dimentional and two-dimentional probabilities densities of the signal at the time moments t, t', and t'' with the proviso that during the interval $t_i \le t \le t_{i+1}$ it is extrapolated by reading x_i ; $\mu=0,1,2,...,M$ is the whole unit of the ratio $|t''-t'|/T_0$; $n=T/T_0$ is the number of readings in realization.

4. MEASUREMENTS OF THE MOMENT CHARACTERISTICS OF RANDOM PROCESSES

By definition of mathematical expectation assessment $<m>$, dispersions $<\sigma^2>$, and correlation function $<R(t''-t')>$ are determined as assessment moments (3) and (4), and are equal [7]:

$$\langle m\rangle = \int_{-\infty}^{+\infty} X\langle \varpi_1[X]\rangle dX = \frac{1}{nT_0}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} m(t|x_i)dt \;; \qquad (5)$$

$$\langle \sigma^2 \rangle = \int_{-\infty}^{+\infty} (X - m)^2 \langle \varpi_1[X]\rangle dX =$$

$$= \frac{1}{nT_0}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} \left\langle [m(t|x_i) - m]^2 + \sigma^2(t|x_i)\right\rangle dt \;; \qquad (6)$$

$$\langle R(t''\text{-}t')\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (X'-m)(X''-m)\langle \varpi_2[X',X'']\rangle dX'dX'' =$$

$$= \langle R(\mu T_0 + \lambda)\rangle = \frac{1}{(n-\mu-1)T_0}\sum_{i=1}^{n-\mu-1}\times$$

$$\times\left\{\int_{t_i}^{t_{i+1}-\lambda} \left\langle [m(t'|x_i)-m][m(t''|x_{i+\mu})-m] + R(t',t''|x_i,x_{i+\mu})\right\rangle dt + \right. \qquad (7)$$

$$\left. + \int_{t_{i+1}-\lambda}^{t_{i+1}} \left\langle [m(t'|x_i)-m][m(t''|x_{i+\mu+1})-m] + R(t',t''|x_i,x_{i+\mu+1})\right\rangle dt \right\},$$

where $m(t|x_i)$ and $\sigma^2(t|x_i)$ are conditional expectation and dispersion of the signal during the interval $t_i \le t \le t_{i+1}$; $R(t', t''| \bullet)$ is conditional correlation function of the signal during the intervals $t_i \le t' \le t_{i+1}$ and $t_{i+\mu} \le t'' \le t_{i+\mu+2}$; $\lambda=|t''-t'|-\mu T_0$, $0\le\lambda\le T_0$.

5. MEASUREMENTS OF SPECTRA IN RANDOM PROCESS

Energetic spectrum $S(\omega)$ of random process is measured by direct application of Fourier transformation and its realization *x(t)* with the following averaging modulus square of the current spectrum $<G(j\omega)>$ [3, 8]. Hence, $<G(j\omega)>$ and $S(\omega)$ in analogue measurements are accordingly equal to

$$< G(j\omega) >= \int_0^T < x(t) > e^{-j\omega t} dt;$$

$$< S(\omega) >= \frac{1}{T} < G(-j\omega) >< G(j\omega) >=$$

$$= \frac{1}{T}\langle G(\omega)\rangle^2,$$

where $j=\sqrt{-1}$ - is affected one; $<G(\omega)>$ - is the value of amplitude – frequency characteristic.

In digital measurements of the ergodic random process and extrapolation of the signal *x(t)* by one previous reading x_i these values will look like

$$< G(j\omega) >= \sum_{i=1}^{n}\int_{t_i}^{t_{i+1}} m(t|x_i) e^{-j\omega t} dt; \qquad (8)$$

$$< S(\omega) >= \frac{1}{nT_0} < G(-j\omega) >< G(j\omega) >. \qquad (9)$$

6. ERRORS OF STATISTIC MEASUREMENTS AND THEIR CHARACTERISTICS

Proximity of the obtained assessments (3), (5)÷(7) to the true probability characteristics of the signal can be evaluated by means of errors. Thus, mathematical expectation $m_{\delta\varpi}$ and correlation function $R_{\delta\varpi}(X',X'')$ of the assessment (3) error are equal to [6]:

$$\left.\begin{aligned} & m_{\delta\varpi}(X) = \langle \varpi_1[X]\rangle - \varpi_1[X]; \\ & R_{\delta\varpi}(X',X'') = \frac{1}{n^2T_0^2}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\{\varpi_2[X',X''|x_i] - \\ & \quad - \varpi_1[X'|x_i]\,\varpi_1[X''|x_i]\}dt'dt'', \end{aligned}\right\} \qquad (10)$$

where $\varpi_1[X]$ is the ideal value of the signal probability density.

In a similar manner, mathematical expectations of assessment (5)÷(7) errors are [3,6,9]:

$$m_{\delta m} = \int_{-\infty}^{+\infty} X\, m_{\delta\varpi}(X)dX = \langle m\rangle - m;$$

$$m_{\delta\sigma^2} = \int_{-\infty}^{+\infty} (X-m)^2 m_{\delta\varpi}(X)dX = \langle\sigma^2\rangle - \sigma^2;$$

$$m_{\delta R}(\mu T_0 + \lambda) = \langle R(\mu T_0 + \lambda)\rangle - R(\mu T_0 + \lambda);$$

$$m_{\delta S}(\omega) = \langle S(\omega)\rangle - S(\omega),$$

where m, σ^2, $R(\mu T_0+\lambda)$ and $S(\omega)$ are ideal values of mathematical expectation, dispersion, correlation function and energetic spectrum of the signal.

Dispersions and correlation functions of assessments (5) ÷ (7),(9) errors are determined as their central moments of the second order and for $\mu \ll n$ are equal to [6] :

$$\sigma^2_{\delta m} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X'X'' R_{\varpi}(X',X'')dX'dX'' = \frac{1}{n^2T_0^2}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} R(t',t''|x_i)dt'dt'' ; \tag{11}$$

$$\sigma^2_{\delta\sigma^2} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (X'-m)^2(X''-m)^2 R_{\varpi}(X',X'')dX'dX'' = \frac{2}{n^2T_0^2}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\{R^2(t',t''|x_i)+2R(t',t''|x_i)\times \times[m(t'|x_i)-m][m(t''|x_i)-m]\}dt'dt'';$$

$$\sigma^2_{\delta R}(\mu T_0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(X'-m)^2(X''-m)^2 R_{\varpi}(X',X'')dX'dX'' = \frac{1}{(n-3\mu)^2T_0^2}\sum_{i=1+\mu}^{n-2\mu}\int_{t_i}^{t_{i+1}}dt''\{\int_{t_{i-\mu}}^{t_{i-\mu+1}}[m(t'+\mu T_0|x_{i+\mu})-m]\times$$
$$\times[m(t''|x_{i-\mu})-m]R(t',t''+\mu T_0|x_{i-\mu})dt' +$$
$$+\int_{t_i}^{t_{i+1}}\langle[m(t'|x_i)-m][m(t'|x_i)-m]R(t'+\mu T_0,t''+\mu T_0|x_i)+$$
$$+[m(t'+\mu T_0|x_{i+\mu})-m][m(t''+\mu T_0|x_{i+\mu})-m]R(t',t''|x_i)+$$
$$+R(t'+\mu T_0,t''+\mu T_0|x_i)R(t',t''|x_i)\rangle dt' +$$
$$+\int_{t_{i+\mu}}^{t_{i+\mu+1}}[m(t'|x_i)-m][m(t''+\mu T_0|x_{i+2\mu})-m]\times$$
$$\times R(t'+\mu T_0,t''|x_{i+\mu})dt'\}; \tag{12}$$

$$R_{\delta R}(\mu' T_0,\mu'' T_0)=\frac{1}{(n-\mu'-2\mu'')^2T_0^2}\sum_{i=1+\mu'}^{n-\mu'-\mu''}\int_{t_i}^{t_{i+1}}dt''\times$$
$$\times\Big\{\int_{t_{i-\mu'}}^{t_{i-\mu'+1}}[m(t'+\mu' T_0|x_{i+\mu'})-m][m(t''|x_{i-\mu'})-m]R(t',t''+\mu'' T_0|x_{i-\mu'})dt'+$$
$$+\int_{t_{i+\mu'-\mu''}}^{t_{i+\mu'-\mu''+1}}[m(t'|x_i)-m][m(t''|x_{i+\mu'-\mu''})-m]R(t'+\mu' T_0,t''+\mu'' T_0|x_{i+\mu'-\mu''})dt'+$$
$$+\int_{t_i}^{t_{i+1}}[m(t'+\mu' T_0|x_{i+\mu'})-m][m(t''+\mu'' T_0|x_{i+\mu'})-m]R(t',t''|x_i)dt'+$$
$$+\int_{t_{i+\mu'}}^{t_{i+\mu'+1}}[m(t'|x_i)-m][m(t''+\mu'' T_0|x_{i+\mu'+\mu'})-m]R(t'+\mu' T_0,t''|x_{i+\mu'})dt'\Big\},$$
$$\mu'\le\mu''; \tag{13}$$

$$R_{\delta G}(j\omega',j\omega'')=\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} R(t',t''|x_i)e^{-j(\omega't'+\omega''t'')dt'dt''};$$

$$\sigma^2_{\delta S}(\omega)=\frac{1}{n^2T_0^2}[R_{\delta G}(j\omega,j\omega)R_{\delta G}(-j\omega,-j\omega)+$$
$$+R_{\delta G}(j\omega,-j\omega)R_{\delta G}(-j\omega,j\omega)+ \tag{14}$$
$$+\langle G(-j\omega)\rangle\langle G(j\omega)\rangle R_{\delta G}(j\omega,-j\omega)+$$
$$+\langle G(-j\omega)\rangle\langle G(-j\omega)\rangle R_{\delta G}(j\omega,j\omega)+$$
$$+\langle G(j\omega)\rangle\langle G(-j\omega)\rangle R_{\delta G}(-j\omega,j\omega)+$$
$$+\langle G(j\omega)\rangle\langle G(j\omega)\rangle R_{\delta G}(-j\omega,-j\omega)].(14)$$

In derivation of the (11) ÷ (14) expressions, the initial moments of the highest orders are expressed by the moments of the first and the second orders. As this is possible for normal distributions only, then expressions (9) ÷ (13) are true for normally distributed processes and errors. For other distribution laws they are considered as approximate ones [7].

7. APPLICATIONS

Integration of the obtained expressions can be considered and analyzed by the concrete example. Let us describe the signal being measured by the stationary random process of uniform probability density

$$\varpi_1[X] = \begin{cases}\dfrac{1}{X_в - X_н}, & X_н \le X \le X_в;\\ 0, & in\ other\ \ cases,\end{cases}$$

with mathematical expectation $m=(X_н+X_в)/2$, dispersion $\sigma^2=(X_в-X_н)^2/12$, correlation function

$$R(\tau) = \begin{cases}\sigma^2, & |\tau| \le \theta;\\ 0, & |\tau| > \theta\end{cases}$$

and energetic spectrum

$$S(\omega) = 2\sigma^2\theta\frac{\sin\ \omega\theta}{\omega\theta},$$

where θ is correlation interval.

Let us also describe the readings errors by the uniform law of distribution with zero mathematical expectation and the dispersion $\sigma^2_\delta=\Delta^2/3$ where 2Δ is quantization step. The errors of the neighbouring readings are considered to be independent. Then conditional probability density in extrapolation by one reading for $t_i \le t \le t_{i+1}$ and $T_0 \le T_к$.

$$\varpi_1[X|x_i] = \begin{cases} \frac{1}{2\Delta}, & x_i - \Delta \le X \le x_i + \Delta; \\ 0, & in\,other\,cases, \end{cases}$$

therefore,

$$m[t|x_i] = \begin{cases} x_i, & t_i \le t \le t_{i+1}; \\ 0, & in\ other\ cases,\ and \end{cases}$$

$$R[\,t', t''|x_i] = \sigma^2(t|x_i) = \sigma^2{}_\delta =$$
$$= \begin{cases} \frac{\Delta^2}{3}, & t_i \le t \le t_{i+1}; \\ 0, & in\ other\ cases\ . \end{cases}$$

By substitution of value (15) into expression (3) and designating as $<P(x_l)>$ the frequency of indication x_j emergence from n readings, we shall obtain conclusively the equation of the classical histogram in the form of [3,6,9] :

$$<\varpi_1[X]> = \begin{cases} \frac{<P(x_l)>}{2\Delta}, & x_1 - \Delta \le X \le x_1 + \Delta; \\ 0, & \text{in other cases.} \end{cases} \quad (15)$$

where l=1,2,...,L, and $L=(X_в-X_н)/2\Delta$.

Mathematical expectation and correlation function of the assessment error are determined by formulae (10) and equal [5] :

$$m_{\delta\varpi}(X) = \frac{1}{2\Delta}[\langle P(x_l)\rangle - \frac{1}{L}];$$
$$R_{\delta\varpi}(X', X'') = \frac{\langle P(x_l)\rangle}{2\Delta n}\{\delta[X'' - X'] - \frac{1}{2\Delta}\},$$

where x_l-Δ ≤ X',X'' ≤ x_l+Δ, l=1,2,…,L.

Assessments $<m>$,$<\sigma^2>$, and $<R(\mu T_0+\lambda)>$, as well as their errors characteristics within the chosen signal model for $\mu<<n$ are equal to :

$$<m> = \frac{1}{n}\sum_{i=1}^{n} x_i;$$

$$m_{\delta m} = <m> - m;$$

$$\sigma^2_{\delta m} = \frac{\sigma^2_\delta}{n};$$

$$<\sigma^2> = \frac{1}{n}\sum^{n} (x_i - m)^2 + \sigma^2_\delta;$$

$$m_{\delta\sigma}{}^2 = <\sigma^2> - \sigma^2;$$

$$<\sigma^2_{\delta\sigma^2}> = \frac{4\sigma^2_\delta}{n}[<\sigma^2> + \frac{\sigma^2_\delta}{5}];$$
$$\langle R(\mu T_0+\lambda)\rangle = \frac{T_0-\lambda}{T_0}\langle R(\mu T_0)\rangle + \frac{\lambda}{T_0}\langle R((\mu+1)T_0)\rangle;$$
$$m_{\delta R}(\mu T_0+\lambda) = \langle R(\mu T_0+\lambda)\rangle - R(\mu T_0+\lambda);$$
$$\sigma^2_{\delta R}(\mu T_0) = \frac{2\sigma^2_\delta}{n-\mu}[\langle\sigma^2\rangle + \langle R(2\mu T_0)\rangle + \frac{\sigma^2_\delta}{5}], \quad (16)$$

$$where \langle R(kT_0)\rangle = \frac{1}{n-k-1}\sum_{i=1}^{n-k-1}(x_i - m)(x_{i+k} - m),$$
$k = \mu,\ \mu+1,\ 2\mu,\ \mu''$-$\mu'$.

Values of spectral characteristics $<G(j\omega)>$, $<S(\omega)>$ and their errors are equal to [10]:

$$\langle G(j\omega)\rangle = \frac{\sin\ \omega T_0/2}{\omega T_0/2} e^{-j\frac{\omega T_0}{2}} T_0 \sum_{i=1}^{n} x_i e^{-j\omega t_i};$$

$$\sigma^2_{\delta G}(j\omega) = \sigma^2_\delta T_0^2 (\frac{\sin\ \omega T_0/2}{\omega T_0/2})^2 \times$$
$$\times \frac{\sin\ \omega T_0 n}{\sin\ \omega T_0} e^{-j\omega(2t_1+T_0 n)};$$

$$\langle S(\omega)\rangle = \frac{1}{n}(\frac{\sin\ \omega T_0/2}{\omega T_0/2})^2 \times$$
$$\times T_0 \sum_{l=1}^{n}\sum_{i=1}^{n} x_l x_i e^{-j\omega(t_i - t_l)};$$

$$m_{\delta S}(\omega) = \frac{1}{n}(\frac{\sin\ \omega T_0/2}{\omega T_0/2})^2 \times$$
$$\times T_0 \sum_{l=1}^{n}\sum_{i=1}^{n} x_l x_i e^{-j\omega(t_i - t_l)} - 2\sigma^2\theta\frac{\sin\ \omega\theta}{\omega\theta};$$

$$\sigma^2_{\delta S}(\omega) = \sigma^2_\delta T_0^2 (\frac{\sin\ \omega T_0/2}{\omega T_0/2})^4 \times$$
$$\times \left\{ \sigma^2_\delta\left[(\frac{\sin\ \omega T_0 n}{n \sin\ \omega T_0})^2 + 1\right] + \right.$$
$$+ \frac{2}{n}\sum_{l=1}^{n}\sum_{i=1}^{n} x_l x_i \cos\ \omega(t_i - t_l) +$$
$$\left. + \frac{\sin\ \omega T_0 n}{n \sin\ \omega T_0}\frac{2}{n}\sum_{l=1}^{n}\sum_{i=1}^{n} x_l x_i \cos\ \omega(t_i + t_l)\right\}.$$

Furthermore, the suggested approach allows to uniquely solve the problem of correlation function $<R(\mu T_0+\lambda)>$ approximation between the discrete values $<R(\mu T_0)>$ and $<R((\mu+1)T_0)>$. With gradational correlation function of signal $R(\tau)$, it will be linear interpolation (16).

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