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COMPUTER AIDED EVALUATION OF MEASUREMENT UNCERTAINTY BY CONVOLUTION OF PROBABILITY DISTRIBUTIONS

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Abstract − The evaluation of measurement uncertainty, from the final user's standpoint, involves several issues which are only partially addressed by the GUM [1]. An evaluation procedure assisted by a computer program can be of great support and provide the user with detailed results useful for further analysis. In this paper a code is presented for the assisted evaluation of measurement uncertainty, based upon the convolution of probability distributions. So the method allows expression of the final result of measurement as a probability distribution, on which it is possible to evaluate useful parameters such as expanded uncertainty, at whatever confidence level. It also permits to evaluate, in probabilistic terms, the risk associated to matching tolerances. Moreover it is possible to manage the uncertainty on dispersion parameters (sometimes called 'uncertainty of uncertainty') considering a modified probability distribution.

In the paper the code is presented together with some case studies showing the support available to the operator, the GUM compatibility and the application importance of the final probability distribution.

Keywords: Measurement uncertainty, Probability distributions, Computer-assisted uncertainty evaluation

1. INTRODUCTION

Research studies on computer aided evaluation of measurement uncertainty and related software codes have been and are currently carried out. Main topics regard the implementation of the GUM procedure, simulation for general purpose or specific complex measuring problems [2-7]. The aim of the software presented in this paper is twofold: to support the user creating a simple model for the measurement procedure; to evaluate the measurement uncertainty on the same hypothesis of the GUM but giving as a result a probability distribution together with values from a strict GUM procedure. The former provide the user with an in depth information regarding the behaviour of his measurement procedure: it is possible to use the final probability distribution to evaluate an expanded uncertainty at whatever coverage level is requested, eventually comparing the results with the classical GUM expanded values. Moreover given an expanded uncertainty at certain coverage level it is possible to use the probability distribution to evaluate the acceptance risk when comparing the measuring result with a tolerance level [8-9].

2. THEORETICAL BACKGROUND

The expression of the result of a measurement by a probability distribution over the set of the possible values of the measurand may be founded over the following considerations.

Let us consider the measurement task consisting in the measurement of a quantity x, with values in X, under the assumption it has a constant value for an interval of time T of interest. The measurement process will include an observation sub-process, giving rise to a vector of observations $\mathbf{y} = [y_1, \ldots y_N]$, produced by a measuring system, under a selected measurement strategy.

The probabilistic relation holding between such observations and the unknown (unobservable) value of the measurand, x, may be expressed, in the most general way, as [9]:

$$p_{\theta}(\mathbf{y}/x) \triangleq p(\mathbf{y}/x;\theta) \qquad (1)$$

where $\theta \in \Theta$ is a vector of parameters representing influence quantities.

Then measurement restitution may be performed according to [9]:

$$p(x/\mathbf{y}) \propto \int_{\Theta} p_{\theta}(y/x) p(\theta) d\theta \; ; \quad x \in X \qquad (2)$$

Formula (2) provides the required foundation for expressing the measurement result as a probability distribution over the set X of the possible values of the measurand. If we now define:

$$\hat{x} \triangleq E_x(x/\mathbf{y}) \qquad (3)$$

it is possible to obtain from formula (3) the following conditional distribution:

$$p(x/\hat{x}) \; ; \quad x, \hat{x} \in X \qquad (4)$$

Incidentally, from a theoretical standpoint, it should be noted that distribution (4), we will be using in the following, actually logically derives from formula (2), which in turn includes the model of the observation process expressed by (1). This metrological background, in our opinion, should not be forgotten, in developing good evaluation procedures.

Let us now consider the following additional hypotheses, conformal the GUM (estimation) model:

1. it possible to express the effect of all influence quantities on the final measurement result as a linear function of the variations in the quantities: $\delta x = \sum_{i=1}^{n} c_i \delta v_i$, with known coefficients c_i;

2. for each of the input quantities, the following information is available:

a) either an estimated standard deviation $\hat{\sigma}_i$, with ν_i degrees of freedom, which may be finite or infinite;

b) or an estimate of the range of variability, say $\left[v_{0i} - \Delta v_i, v_{0i} + \Delta v_i\right]$, with limit either certain or subject to a relative uncertainty, say $\alpha_i = \Delta\Delta v_i / \Delta v_i$;

3. it is possible to identify a subset of mutually uncorrelated input quantities, say, without loss of generality, $v' = \left[v_1, v_2, ..., v_m\right]$, $0 \le m \le n$, for which the corresponding correlation matrix is known $R = \left[r_{ij}\right]$.

Under the assumed hypotheses, we have:

$$p\left(x/\hat{x}\right) = p_{\Delta\mathbf{v}}\left(x - \hat{x}\right) \quad (5)$$

where $p_{\Delta\mathbf{v}}(x)$ may be expressed as the composition, through a convolution rule, of the distribution accounting for the global influence of the subset of uncorrelated input quantities (if not null), $p_{\Delta\mathbf{v},uncorr}(x)$, with the distribution accounting for the effect of complementary subset of mutually correlated input quantities (if not null), $p_{\Delta\mathbf{v},corr}(x)$.

So we have:

$$p_{\Delta\mathbf{v}}(x) = p_{\Delta\mathbf{v},uncorr}(x) * p_{\Delta\mathbf{v},corr}(x) = \int p_{\Delta\mathbf{v},uncorr}(x-\xi)\, p_{\Delta\mathbf{v},corr}(\xi)\, d\xi \quad (6)$$

The numerical calculation of such distribution will be detailed in the next paragraphs.

3. THE ASSISTED EVALUATION PROCEDURE

The user who needs to evaluate a measurement uncertainty has to face some difficulties form the clear identification of the devices used in the measuring chain to the collection of data and finally computational problems to evaluate the final uncertainty.

The proposed code is based on a schematisation of the measuring chain as a sequence of standard blocks each representing a component of the measuring chain. Each block is characterized by a set of properties that describes the overall behaviour of the component as an element of the measuring chain and moreover as a contributor to the measuring uncertainty. In this phase of development the block represents a linear transfer function with uncertainty on the sensitivity and on the offset: a proper combination of blocks can provide a model even for a complex measuring procedure.

Each influence quantity contributing to the overall uncertainty is characterized by its proper dispersion parameter, by the kind of underlying probability distribution, by the number of degrees of freedom or other quantifier of the uncertainty on dispersion parameters. For mutually correlated quantise, the correlation coefficient must also be specified. Such information are also conformal to the requirements of the GUM. All the information referring to each block in the measuring chain is treated as defining an object, in the object oriented environment in which the code is developed.

The evaluation of the standard and expanded uncertainty proceeds according to two independent methods: the GUM recommended evaluation principles, and the convolution of the probability densities of each contributor.

3.1. Available options

The Assisted procedure can manage several different cases, schematically it is possible to distinguish:

i- type A or type B uncertainty values;

ii- correlated and uncorrelated quantities and mixed combinations of them;

iii- presence of uncertainty on the dispersion parameters of the quantity.

i - Type A evaluations are treated with the proper degrees of freedom using t-Student distributions;

type B evaluations are treated with probability densities such as uniform, triangular and other, with a proper dispersion parameter.

ii – Correlated quantities are first of all combined as specified by the GUM and then they are associated to a normal distribution that is considered as another uncorrelated quantity. If they present an uncertainty on the dispersion parameter, their probability densities are previously computed according to (iii), then their new dispersion figures are computed and combined as here described.

iii – The uncertainty on the dispersion parameter is managed by associating to the quantity a proper probability distribution. The new probability distribution depends on the original shape: in case of normal PD a t Student is used.

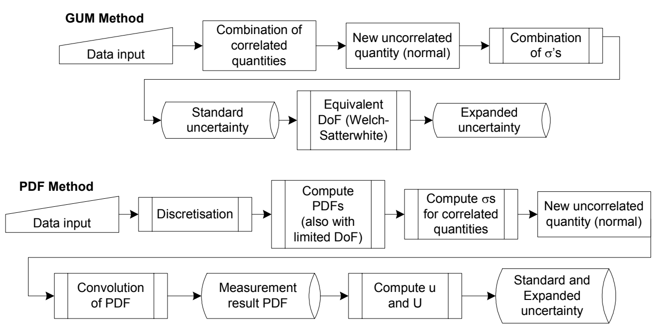

Fig. 1. Schematic flowchart for the codes implementing the GUM and the PD methods

In case of uniform PD with uncertainty on the distribution limits, then a new PD is assigned that is the mean among all the possible probability densities considering the uncertainty on the distribution parameter.

Uncorrelated probability densities are then convolved together giving the final probability distribution for the considered measurement process. It is then possible to evaluate directly both the standard and the expanded uncertainty, U, given the desired coverage level, p_0. Note that having the probability distribution, there is no need for the application of the Welch-Satterwaite formula [10-11]. The codes provides also standard and expanded uncertainties computed strictly according to the GUM. As described in the next section, at the present development stage the assisted procedure consists of two separate software codes for the implementation of the GUM and the PD methods, their flowcharts being presented in figure 1.

3.2. The software development

Particular care was dedicated to the software development procedure, integrating this activity in the quality management policy of the laboratory.

The development followed two separate lines, one dedicated to GUM methods, the other to probability distributions methods. Successive versions of the software proceeded in a parallel way integrating at each step a limited set of new capabilities. In such a way at each step a complete software validation was possible together with a direct comparison of the results obtained with the two methods at the same development stage. The sequence of development for both methods is presented in table 1 and at the last stage covers all the available options described before.

TABLE I. Development steps and their functionalities

Stage	Functionalities
0	Only uncorrelated quantities, infinite degrees of freedom (DoF), no uncertainty on the parameter.
1	Only uncorrelated quantities, infinite or limited DoF, uncertainty on the parameter.
2	Correlated or uncorrelated quantities, infinite or limited DoF, uncertainty on the parameter also for correlated quantities.

At each step of development a detailed documentation was produced including main functionalities, software strategies, user manual and discussion on validation policies and results. The code and the corresponding documentation are then inserted in the laboratory archive and managed according to the quality procedures of the lab.

The software is written in the Matlab® environment according to an object oriented approach that is particularly suited to our model of measurement chain and to the assisted procedure [12]. The memorisation of the influence quantities in an object form proved to be really effective in order to have all the characteristics of a quantity, or a block in the model, available at the same time. Each version of the software consists of a set of separate codes or functions, each one dedicated to a specific aim. In such a way it is possible to re-use functions whenever necessary in new software versions. Moreover a separate validation of each function is often much more feasible and efficient than a complete validation of the overall software package.

The assisted procedure is intended to have an active place in the measurement procedure, in order to give a figure of the reliability of the measurement result. So the assisted procedure has to be treated with metrological criteria and validation according to reference cases is necessary. The validation policy was selected according to the different functionalities. In particular for the overall code a set of examples was selected from GUM appendix as reference cases. Moreover, in order to test completely all the available options, a set of more complex and less common cases was defined. Validation was considered positive when besides the fundamental functionalities of the code such as data input and output, the results were consistent with the expected results obtained by manual computation according to the GUM, to the PD method or obtained directly from the GUM appendix [13].

In order to have an easier validation the code was developed in different phases according to its possible functionalities. In the first phase, only discrete probability densities have been considered to avoid the usual problems in going from a continuous domain to a discrete one. Then a discretisation procedure was set up and validated, extending the software capabilities to the continuous case.

Quantisation was deeply investigated, as it is a key-point regarding the accuracy of the computation. This is moreover important when small differences due for example to the effect of the uncertainty of the dispersion parameter have to be evaluated.

The policy of discretisation considers the use of an odd number of points over the full interval. The number is determined on half range and then extended to the full range adding the central point.

When changing from a continuous to a discrete probability distribution there may be two main effects due to the discretisation policy: - the position of the point on the edges can produce a discrete distribution slightly larger or smaller than the original distribution; - the truncation of an infinite probability distribution in particular a normal, to a given coverage level. A study was carried out to investigate the optimal discretisation for normal and uniform distribution in order to have an acceptable computational accuracy together with an acceptable computation time. A further study will regard the Student and the uniform with uncertainty on the limits distributions.

Some care was dedicated to the design and realisation of the user interface, which assists the user in the organization of the input data necessary for the computation [14].

At the end of the data input phase a check procedure is provided in which data are verified both by the user and partially by the supporting software.

4. TEST CASES

Test cases are presented in order to show the possibilities offered by the considered method and the differences with the GUM procedure. The effect of the uncertainty on the dispersion parameter is shown together with the effective calculation of expanded uncertainties. Test cases include a GUM example in order to have direct comparison of the results and direct validation of the figures evaluated according to standard methods; a complex case to show all the available features and a discrete case to show how it is possible to manage discrete measurement systems.

4.1. An example from the GUM

This first test case is the example H1 of the GUM and regards the measurement of standard reference blocks.

Uncertainty evaluation is carried out after linearization the measurement model according to the GUM procedure, and table 2 presents the input data.

TABLE II. Input data for test case 4.1

Quantity	PD	Dispersion parameter	DoF/Uncertainty on parameter
1	Normal	25	18
2	Normal	9.7	26
3	Uniform	2.9	10% range or 50 DoF.
4	Uniform	16.6	50% range or 2 DoF

Figure 2 presents the PDs of this example. Note that:
- a t-Student PD is used for the first two quantities; - the uniform distributions are modified according to the uncertainty on the dispersion parameter.

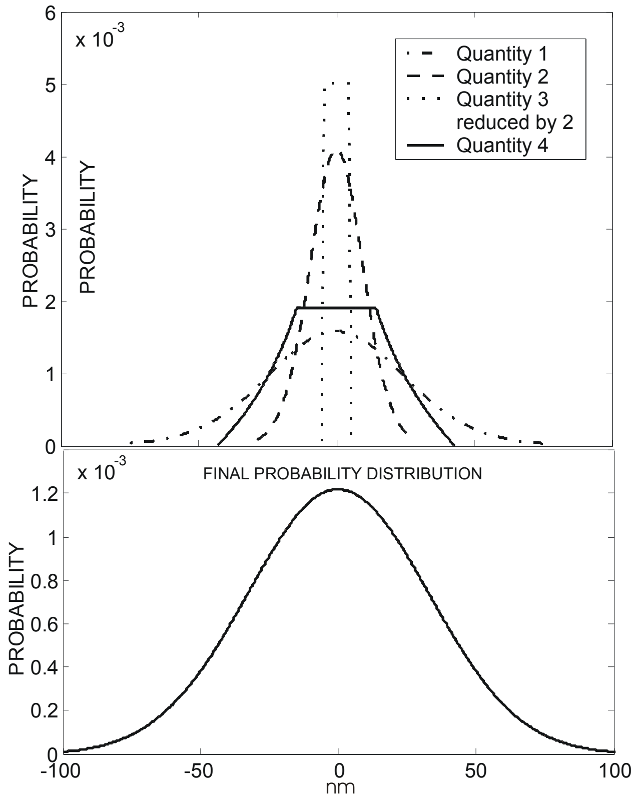

Fig. 2. Test case 1: probability distributions of the input quantities and convolved final distribution on a separate scale.

Table 3 presents a comparison of the results, in which it is possible to appreciate that the evaluation of the expanded uncertainty based on the final PD is slightly different from the figures based upon the GUM method, which are based on the Welch-Satterwaite approximation.

TABLE III. Test case 3.1, comparison of GUM and PD results

	GUM	PD
Standard uncertainty, u [nm].	32	32
Expanded uncertainty, 95%, U_{95} [nm].	68	64
Expanded uncertainty, 99%, U_{99} [nm].	93	82

4.2. Overall capabilities test case

In this test case all the available options from the procedure are considered.

TABLE IV. Input data for test case 4.2

	PD	Dispersion parameter	DoF/Uncertainty on parameter	Correlation coefficient
1	Uniform	1.4	20% range or 25 DoF.	-0,5
2	Normal	1.1	5	
3	Uniform	1.7	10% range or 50 DoF	-
4	Normal	0.6	6	-

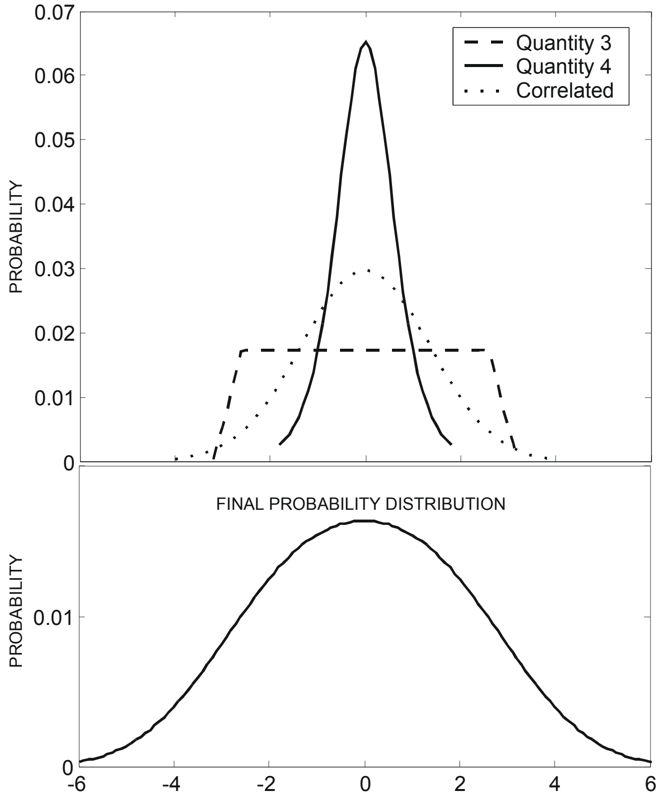

Fig. 3. Test case 2:probability distributions of the input quantities and convolved final distribution on a separate scale.

In this case the normal probability distribution for the correlated quantities has a dispersion parameter computed according to the GUM, but considering the actual dispersions of the input distributions, modified to take into account the uncertainty on their distribution parameters.

There are either correlated and uncorrelated quantities which present limited DOF or uncertainty on the dispersion parameter. Input data are presented in table 4. Input and final distributions are presented in figure 3.

TABLE V. Test case 3.2, PD results (arbitrary units)

	PD
Standard uncertainty, u	2.2
Expanded uncertainty, 95%, U_{95}	4.2
Expanded uncertainty, 99%, U_{99}	5.4

Final figures are presented in table 5. Since the PD method evaluates the correlated quantities distribution not according to the GUM a comparison of the final figures is unfeasible.

4.3. Discrete test case

In this last example a discrete case is considered in order to show that approximating a discrete PD by a continuous one can lead to misleading results.

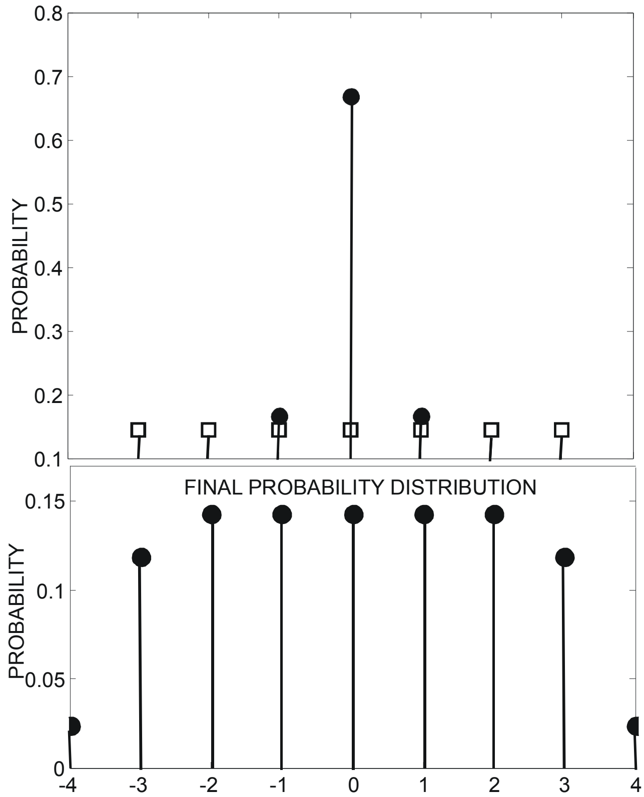

Fig. 4. Test case 3: probability distributions of the input quantities and convolved final distribution on a separate scale.

This is the case in which measurement results are affected for example by a large quantisation effect, in such a way that the repeated measurement distribution appears as discrete and not continuous.

Figure 4 presents the input and output discrete PDs, together with the final one. Table 7 presents a comparison of the final results.

TABLE VI. Input data for example 3.3

	PD	Dispersion parameter	DOF/Uncertainty on parameter
1	Uniform	3	-
2	Discrete Normal (discrete t-Student)	0.5	10

Note that since the final PD is obtained as a result of a convolution process between two discrete distributions, its shape is discrete itself. In this case the use of a continuous PD such as a t-Student for the evaluation of the expanded uncertainty is not a proper choice and the evaluation based upon the final PD is the only possible way of proceeding.

TABLE VII. Test case 3.3, comparison of continuous approximation and PD results.

	GUM	PD
Standard uncertainty, u	1.8	2.1
Expanded uncertainty, 95%, U_{95}	3.5	3
Expanded uncertainty, 99%, U_{99}	4.7	4

In table 7 some figures are presented to compare the continuous approximation and calculation of discrete distribution. Note that while the expanded uncertainty evaluated by the GUM method gives figures that do not consider the discrete resolution of the instrument, the proposed method gives a result compatible with the instrument's resolution, since it evaluates the expanded uncertainty directly on the final discrete probability distribution.

5. CONCLUSIONS

A software for the evaluation of measurement uncertainty using probability distributions was presented, together with a set of test cases to present its functionality. The use of the proposed method and assisted procedure gives effective results to the user and additional information regarding the final probability distribution. It is also possible apply the standard GUM method. Computational efficiency is ensured by a careful discretisation together with nowadays powerful PC. Data input is guided by a proper and friendly user interface.

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