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A GENERALIZED PROCEDURE FOR MODELLING OF MEASUREMENTS FOR EVALUATING THE MEASUREMENT UNCERTAINTY

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Abstract – The modelling of the measurement is a key element of the evaluation of measurement uncertainty in accordance with the basic concept of the *Guide to the Expression of Uncertainty in Measurement (GUM).* The model equation expresses the relationship between the measurand and all relevant *input quantities* contributing to the measurement result. It serves as a basis for propagation of the *probability density distributions* for the *input quantities* or, in case of (almost) linear systems, for *Gaussian* propagation of the related standard uncertainty contributions.

A practical and highly versatile modelling concept has been developed. It is based on both the idea of the classical *measuring chain* and the *measurement method* used. Therefore, this concept gets on with only a few generic structures.

The concept has led to a modelling procedure which is structured into five elementary steps. Only three types of modelling components are employed. It holds for most kinds of measurements performed in the steady state.

Keywords: probability, uncertainty, modelling.

1. INTRODUCTION

In accordance with the *GUM* [1], it is the aim of the modelling procedure to mathematically establish the relationship between the measurand Y and all relevant (random) *input quantities* $X_1,\ldots, X_N$ which may contribute to the uncertainty associated with the measurement result:

$$Y = f(X_1, X_2, ..., X_N) \tag{1.1}$$

This relationship serves as a basis for propagation of the *probability density distributions* for the *input quantities* or, in case of linear or quasi-linear systems, for *Gaussian* propagation of the standard uncertainty contributions attributed to the *input quantities*. Furthermore, the model equation opens up an easy way to implement computer-aided uncertainty budgeting.

But neither the *GUM* nor other relevant uncertainty documents provide a systematic and generally applicable modelling procedure. Therefore, to practitioners, modelling appears to be the most difficult problem in uncertainty evaluation.

First approaches to a consistent modelling procedure have been made by *Bachmair* [2], *Kessel* [3], *Kind* [4] and by a joint working group of the *Physikalisch-Technische Bundesanstalt (PTB)* and the German Standardization Organization *DIN* [5].

As the above approaches, the modelling procedure presented is based on both the idea of the *measuring chain* and the *measurement method* used. It is the aim of this work to present a straightforward and highly versatile operational procedure for the modelling of measurements in the steady state that is applicable to most areas of uncertainty evaluations performed.

2. GUM CONCEPT

2.1 Basic relationships

The GUM concept for evaluating the uncertainty is based on the knowledge about the measuring process and the quantities which may contribute to the measurement result and its associated uncertainty. Therefore, the starting point of uncertainty evaluation is to take up and gain this knowledge.

In accordance with the *GUM* concept (see *GUM*, clause 3.3.4) , the (always being incomplete) knowledge about each contributing (random) *input quantity* X_i is to be expressed by means of *probability density functions (pdf)* $\varphi_{Xi}(\xi_i)$. The expectation value of the *pdf* is the best estimate of the value of the quantity (see equation (2.1)) and its standard deviation is the uncertainty u_{xi} associated with this estimate (see equation (2.2)):

$$x_i = E\left[X_i\right] = \int_{-\infty}^{+\infty} \varphi_{Xi}(\xi_i)\xi_i \, d\xi_i \tag{2.1}$$

where ξ_i are the possible values of the quantity X_i.

$$u_{xi} = \left\{E\left[\left(X_i - x_i\right)^2\right]\right\}^{1/2} = \left\{\int_{-\infty}^{+\infty} \varphi_{Xi}\left(\xi_i\right)\left(\xi_i - x_i\right)^2 d\xi_i\right\}^{1/2} \tag{2.2}$$

But how to obtain an appropriate *pdf* that reasonably reflects the existing (incomplete) knowledge about a quantity? It can be obtained by utilizing the *principle of maximum information entropy (pme)* [6], that, for example, yields

- a *rectangular pdf* if one knows that the values ξ_i of the quantity X_i are contained in an interval (practical examples: given tolerances or error limits, digital resolution),
- a *Gaussian (normal) pdf* if one knows the best estimate $x_i = \mathrm{E}[X_j]$ and the associated standard uncertainty u_{xj} of the quantity X_j (practical examples: statement of a calibration result, result of a statistical analysis expressed by a mean and a standard deviation).

If new information is available, the change of a given *pdf* is described by *Bayes´ theorem* [7][8]: The *posterior pdf* $\varphi_X(\xi|D,I)$ taking account of new data D results from the *prior pdf* $\varphi_X(\xi|I)$ taking account of prior information I as product of a constant C, the *Likelihood* $l(\xi|D,I)$ and the *prior pdf*:

$$\varphi_X(\xi\,|D,I)d\xi = Cl(\xi\,|I)d\xi \tag{2.3}$$

C follows from the normalization of the *posterior pdf*.

The *pdf* for the measurand Y, $\varphi_Y(\eta)$ is given by the integral

$$\begin{aligned}\varphi_Y(\eta) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\varphi_{X1,\ldots,XN}(\xi_{1,}\ldots\xi_N)\ldots \\ \ldots\delta(\eta - f_Y(\xi_{1,}\ldots\xi_N))d\xi_1\ldots d\xi_N\end{aligned} \tag{2.4}$$

where f_Y is the functional relationship of the values of the involved quantities and the value of the measurand. From the above *pdf* $\varphi_Y(\eta)$, the expectation value of the measurand $y = \mathrm{E}[Y]$ and its associated uncertainty u_y can be derived:

$$y = \int_{-\infty}^{\infty}\varphi_Y(\eta)\eta\,\mathrm{d}\eta \tag{2.5}$$

$$u_Y = \left\{\int_{-\infty}^{\infty}\varphi_Y(\eta)(y-\eta)^2\,\mathrm{d}\eta\right\}^{1/2} \tag{2.6}$$

The way of uncertainty determination by means of *pdf propagation* is illustrated in Fig. 1.

But because equation (2.4) can be analytically computed only in fairly simple cases, modern uncertainty evaluation uses the *Monte-Carlo Method* as an integration technique [8].

2.2 Monte-Carlo Method

The *Monte-Carlo Method* is an integration technique which inter alia is utilized for propagation of *pdf*s [9] (Fig. 1).

It is based on the sampling of the cumulative input *pdf*s: With uniform probability, probability values $\Phi_{Xi}(\xi_i)$ of the involved input quantities are to be selceted. Then, the related arguments ξ_i of each quantity are to be combined in accordance with the model equation $Y=f(X_1, \ldots, X_N)$ yielding values η_i. From a sufficient number of samples, a frequency distribution is obtained that approximates the probability distribution $\varphi_Y(\eta)$ which may be attributed to the measurand Y. Fig. 2 illustrates these techniques.

This Monte Carlo integration technique (see Fig. 2) converges always if the variance of *pdf* for the output quantity is finite. However, this is always the case in meaningfully designed measurements. The technique handles linear

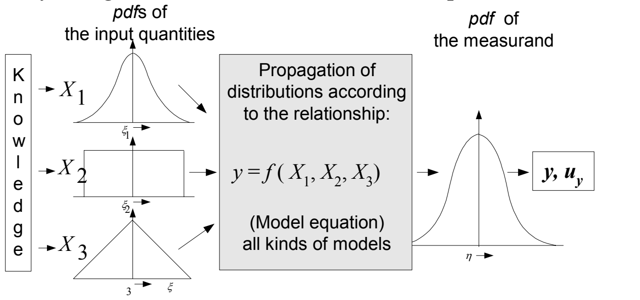

Fig. 1: Illustration of the concept of *pdf* propagation in uncertainty evaluation. symbols: Y-measurand; X_1, X_2, X_3 - input quantities contributing to the result of the measurement; $y = \mathrm{E}[Y]$; u_y - standard uncertainty associated with y

and non-linear models alike whereas the expansion of the law of uncertainty propagation (equation 2.8) to higher orders (see *GUM* [1], note to clause 5.1.2) may require substantial additional effort. To practitioners, however, the *Monte-Carlo Method* still appears to be “very mathematical” but not tangible enough.

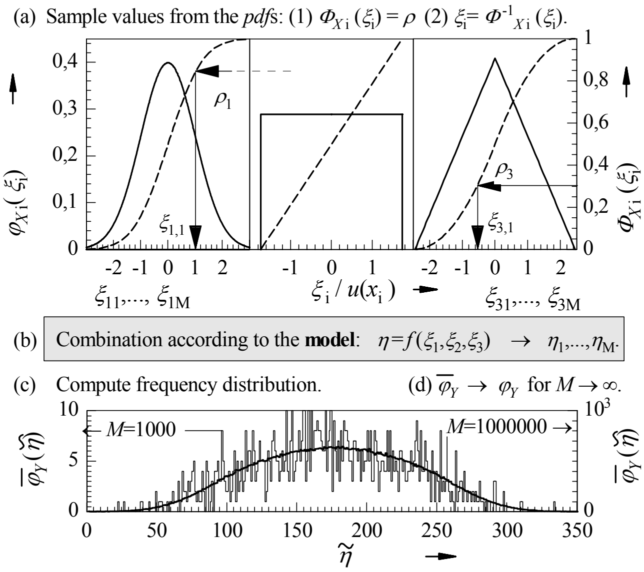

Fig. 2: Visualisation of *Monte-Carlo* integration. (a) The upper three graphs show (solid lines) the *pdfs* to the (input) quantities and (dashed lines) the respective cumulative distributions. The *arrows* illustrate the sampling from cumulative distributions. (b) Combination of the arguments. (c) The curves in the bottom part show computed frequency distributions for two sample sizes M. (d) Approximation of the *pdf* to the measurand φ_Y.

2.3 Standard GUM Method

During the last decade, the *Standard GUM Method* [1] has become a worldwide-recognized standard for the evaluation of measurement uncertainty. In difference to the *Monte-Carlo Method*, the *Standard GUM Method* is based on the *Gaussian* uncertainty propagation.

The *model equation* (see equation (1)) together with the estimated values x_i serve as a basis for the determination of the expectation value of the measurand:

$$y = f(x_1, x_2, \dots, x_N) \quad (2.7)$$

The uncertainty propagation is based on the rules of *Gaussian error propagation* and first-order *Taylor series expansion* assuming that, at least in narrow ranges around the expectation values (operating points), the partial responses of the output quantity to changes of the respective input quantities may be described sufficiently by their first partial derivatives $\partial f / \partial x_i |x_i$ (see also 3.1):

$$u_y = \left\{ \sum_{i=1}^{N} \left(\frac{\delta f}{\delta x_i} \right)^2 u_{xi}^2 + 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} u_{xixj} \right\}^{1/2} \quad (2.8)$$

where $u_{xixj} = u_{xi} \cdot u_{xj} \cdot r(x_i; x_j)$ is the estimated covariance of the quantities X_i and X_j, $r(x_i, x_j)$ being the correlation coefficient.

Due to the use of first-order *Taylor series expansions*, the application of the *Standard GUM Method* is limited to linear or quasi-linear systems. Fig. 3 illustrates the concept of *Gaussian* uncertainty propagation in uncertainty evaluation.

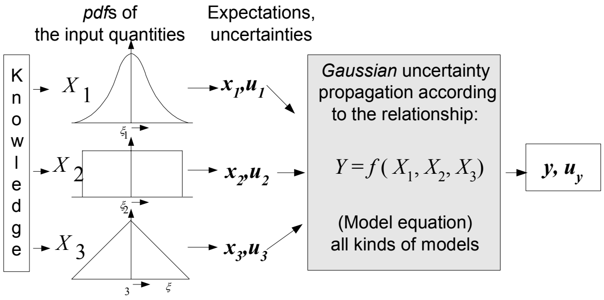

Fig. 3: Illustration of the concept of *Gaussian* uncertainty propagation

3 MODELLING PROCEDURE

3.1 Concept

The modelling of the measurement is a key element of uncertainty evaluation in accordance with the *GUM* (see 2.2 and 2.3).

The modelling concept presented [10] is intended for the modelling of measurements that are performed in the steady state. The concept is based on both the idea of the classical *measuring chain* [4] [5] and the *method of measurement* [4] used.

The *measuring chain* constitutes the path of the measurement signal from *cause* to *effect*. The measuring system or the measuring process is regarded as a series of non-reactive functional elements (or a sequence of operational steps) to carry out the measurement. In metrological practice, the following assumptions can be made:

- The great majority of measuring systems and devices (elements) may be regarded to have linear characteristics or, at least in narrow ranges around the operating point, a linear characteristic may be assumed.
- The steady-state characteristic of a measuring system is always related to well adjusted and known operating conditions (idea of the fictitious *ideal measurement*).
- The "real world of measurement" may be taken into consideration by means of *deviations* of the real influence quantities and other parameters from the above mentioned well adjusted and known conditions.

On the above assumptions, in steady state, almost all functional elements or operational steps of a measuring system or process may be described by an approximately constant transmission factor and by *deviations* representing the imperfections of the measurement. *Deviations* may have impact on transmission factors and they may result in offsets of the outputs. Fig. 4 illustrates this concept of a *disturbed ideal element* that can mathematically be expressed by the following relationship.

$$X_{KOUT} = X_{kIN}(G_{OK} + \delta G_{OK}) + \delta Z_K \quad (3.1)$$

where: X_{kIN} - (random) quantity acting on the input of the element k; X_{KOUT} - (random) quantity at the output of the element k; G_{OK} - (steady-state) transmission factor of the element k which depends on the chosen operating point; δG_{OK} - parameter deviation which results in a deformation of the characteristic; δZ_K - parameter deviation which results in an offset of the output.

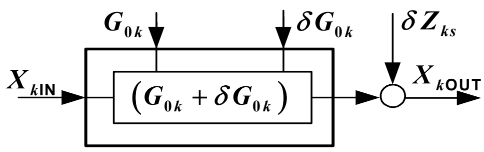

Fig. 4: Concept of an "disturbed ideal" transmission element

The above concept does include the case of the mathematical combination of (random) quantities, e.g. by multiplication of a resistance with a current in order to obtain the voltage across this resistance. In order to ensure the linear response characteristic, the expectation values of the additional input quantities are required to be constant, $E[X_{KIN2}]$= constant.

3.2 Modelling components used

For schematic depiction of the *cause-and-effect relationship* of the measurement, only three types of components are employed:

(1) Parameter sources (SRC): provide or reproduce a measurable quantities, e.g. the measurand (see Fig. 5).

(2) Transforming units (TRANS): represent any kind of parameter processing and influencing (see Fig. 6).

(3) Indicating units IND: indicate their input quantities (see Fig. 7).

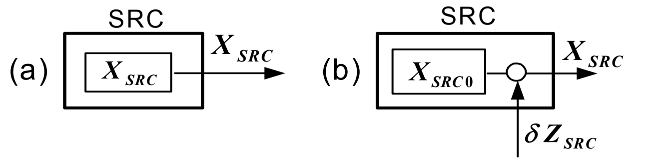

Fig. 5: Graphical scheme of a parameter source. (a) general case when providing a measurable quantity; (b) material measure; symbols: X_{SRC} – (random) quantity provided by the source; δZ_{SRC} - deviation due to the imperfection of the component quantities, X_{TOUT} – output quantity; δZ_{PT} – superimposing deviation, e.g. due to the susceptibility of the unit to external conditions $(\underline{P})$

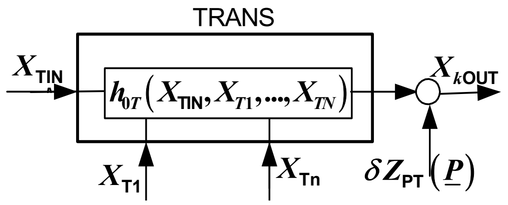

Fig. 6: Graphical scheme of a transforming unit. symbols: h_{0T} - steady-state response function which defines the parameter processing (e.g. combination of quantities); X_{TIN} – primary input quantity; X_{Tj} – additional input quantities

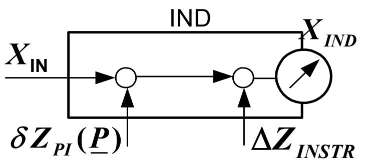

Fig. 7: Graphical scheme of an indicating unit. X_{IN} – input quantity of the unit; symbols: X_{IND} – indicated quantity; ΔZ_{INSTR} – instrumental error; δZ_{PI} – deviation due to the susceptibility of the unit to external conditions $\underline{P}$

3.3 Modelling procedure

The modelling procedure consists of five elementary steps:

1st step: Description of the measurement; identification of the *causal quantity* and of the *measurand*; identification of the *measurement method* used.

2nd step: Setting up schematically *the cause-and-effect relationship* of the (fictitious) *ideal measurement* (Fig. 8).

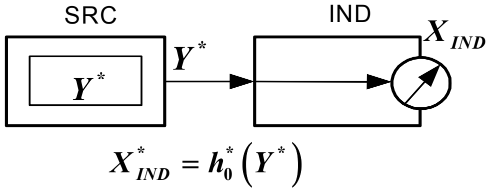

Fig. 8: Example: Graphical depiction of the *cause-and-effect relationship* of a (fictitious) *ideal measurement*

3rd step: Inclusion of all imperfections, influences and effects of incomplete known parameters; insertion into the *cause-and-effect relationship* by means of *deviations* from the above *ideal measurement* (Fig. 9); expressing the resulting *cause-and-effect relationship* in mathematical terms, e.g. (see Fig. 9) $X_{IND} = h_0(Y_0, \delta Z_{SRC}, \delta Z_T, \Delta Z_{INSTR}, \delta X_{IND})$.

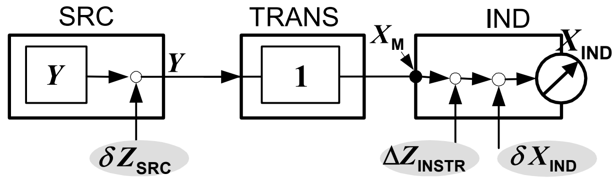

Fig. 9: Example: Graphical depiction of the *cause-and-effect relationship* of a real measurement

4th step: Identification and consideration of possible correlations:

- Consideration of functional dependencies by introducing them into the *cause-and-effect relationship* of the real measurement (see 3rd step), or
- Taking (estimated or experimentally determined) correlation coefficients into account when propagating the uncertainty contributions (see equation (2.8)).

5th step: Conversion of the *cause-and-effect relationship* into the (steady-state) model equation, e.g. (see 3rd step) $Y = f_0(X_{IND}, \delta Z_{SRC}, \delta_{ZT}, \Delta Z_{INSTR}, \delta X_{IND})$, or generally expressed:

$$Y = h_0^{-1}(X_1, X_2, ..., X_N) = f_0(X_1, X_2, ..., X_N) \quad (3.2).$$

3.4 Role of the measurement method used

The structure and the chaining sequence of he *cause-and-effect relationship* are determined by the *method of measurement* used.

Direct measurements result in a un-branched chain of the components used. The generic structure of the *cause-and-effect relationship* of a *direct measurement* is shown in Fig. 10.

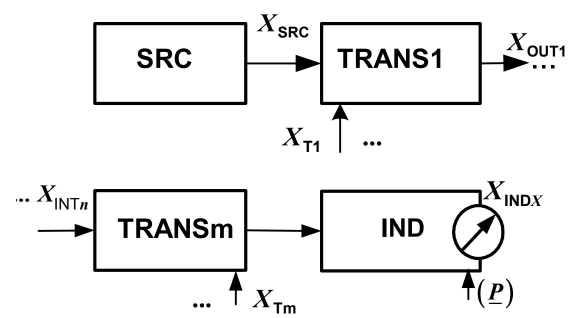

Fig. 10: Generic structure of the *cause-and-effect relationship* of a *direct measurement*. symbols: X_{T1},.., X_{Tn} – additional input quantities; X_{SRC} – quantity provided by the source; X_{IND} – indicated quantities; ΔZ_{INSTR} – instrumental error; P – external conditions, other symbols see 3.2

Other methods are used to achieve high accuracies and to ensure traceability of calibration results. These methods mostly result in branched *cause-and-effect relationship*.

Examples are given with the direct comparison of two indicating measuring instruments and with the substitution method.

Fig. 11 and 12 show the generic structures of the *cause-and-effect relationships* of these two methods. When deriving the mathematical *cause-and-effect relationships* from block diagrams having branched structures, e.g. the above methods, for each branch a separate (partial) equation is to be set up (see 4).

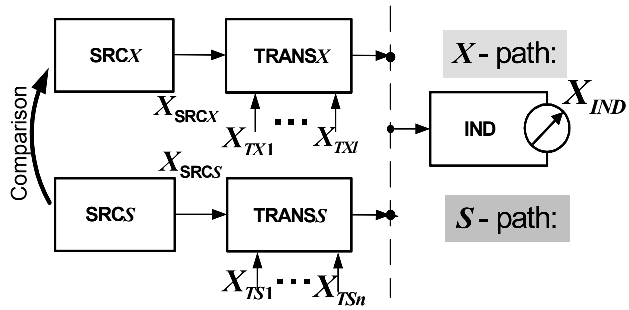

Fig. 11: Generic structure of the *cause-and-effect relationship* of a direct comparison of indicating measuring instruments. symbols: TRANS*X* – transforming unit of the *X*-path; TRANS*S* – reference transforming unit; IND*X* – indicating unit under test; IND*S* – indicating standard; other symbols see Fig. 10.

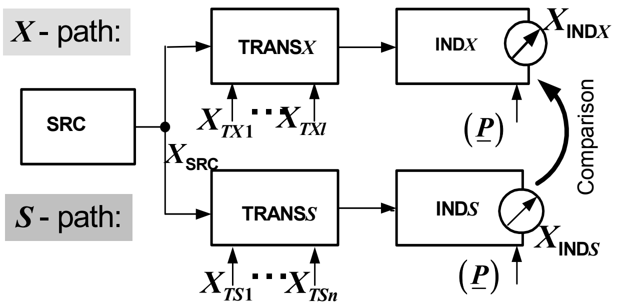

Fig. 12: Generic structure of the *cause-and-effect relationship* of a measurement using the substitution method. symbols: SRC*X* – material measure under test; SRC*S* – standard (material measure); TRANS*X* – transforming unit of the *X*-path; TRANS*S* – transforming unit of the *S*-path. IND – comparator, other symbols see Fig. 10

From the mathematical *cause-and-effect relationship*, the model equation may be derived. In case of block diagrams with branched structures, influences and imperfections of the commonly used path can be neglected.

EXAMPLE

4.1 Modelling procedure

The modelling procedure is explained with the calibration of a liquid-in-glass thermometer. The example has been simplified.

1st step:

- *Description of the measurement:* A mercury-in-glass thermometer is to be calibrated in steady state at 20 °C. Together with a standard thermometer, the instrument to be calibrated is immersed in a thermostatted and stirred water bath (see Fig. 13).
- *Causal quantity:* Temperature of the water bath, t_{Bath}.
- *Measurand:* Instrumental error Δt_x of the thermometer to be calibrated at the 20 °C scale value.
- *Measurement method:* Direct comparison of two indicating instruments.

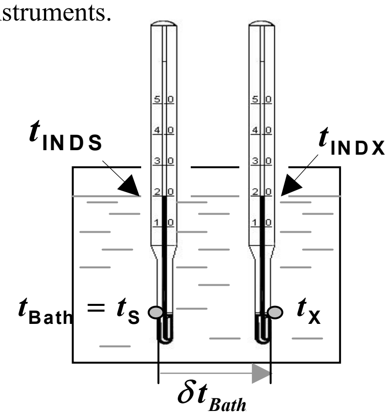

Fig. 13: Example: Calibration of a liquid-in-glass thermometer in a water bath. t_S – temperature of the standard; t_x – temperature of the thermometer under test; t_{INDX}, t_{INDS} indicated temperatures

2nd step:

Fig. 14 shows the block diagram of the *cause-and-effect relationship* of the (fictitious) *ideal measurement*.

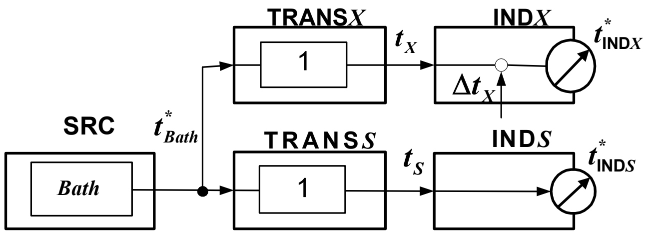

Fig. 14: *Cause-and-effect relationship* of the *ideal measurement* according to the chosen example. SRC - thermostatted bath; TRANSX, TRANSS – temperature gradients in the bath, INDX - thermometer to be calibrated; INDS - standard thermometer; other symbols see text

3rd step:

Fig. 15 shows the *cause-and-effect relationship* of the real measurement. The following imperfections have been included: Δt_S instrumental error of the standard (known and unknown contributions); δt_{BathX} deviation of the temperature of the Instrument to be calibrated from bath temperature t_{Bath} that is assumed to be equal to the temperature of the standard. The quantities t_{INDX} and t_{INDS} are taken from repeated reading of the thermometers. In mathematical terms, the cause-and-effect relationship of the real measurement reads:

- X-path: $t_{INDX}=t_{Bath}+\delta t_{BathX}+\Delta t_X$
- S-path: $t_{INDS}=t_{Bath}+\Delta t_S$

4th step: *(Inclusion and consideration of correlations)*

For the sake of simplification, all involved quantities, parameters and observations are assumed to be independent of each other.

5th step:

From the above *cause-and-effect relationship* of the real measurement, the following model equation is obtained:

- $\Delta t_x = t_{INDX} - t_{INDS} + \Delta t_S - \delta t_{BathX}$

4.2 Evaluating the measurement uncertainty

Due to the linear *model equation* of the chosen example (see 4.1), the *Standard GUM Method* may be used to evaluate the measurement uncertainty. After modelling the measurement, the most important step is to evaluate the involved quantities t_{INDX}, t_{INDS}, Δt_S and δt_{BathX}. To each of these quantities an expectation value and an associated standard uncertainty are to be assigned: t_{INDX} and t_{INDS} may be evaluated by statistical analysis of series´of repeated observations (method *type-A*). The knowledge about Δt_S should be taken from the calibration certificate of the standard. δt_{BathX} may be estimated from the manufacturer´s information sheet about the bath (method *type-B*). Table 1 gives an example of such an evaluation.

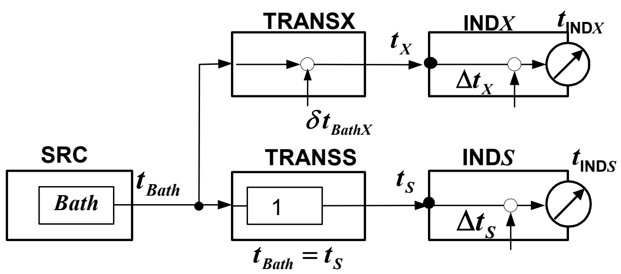

Fig. 15: C*ause-and-effect relationship* of the real measurement according to the chosen example. SRC - thermostatted bath; TRANSX, TRANSS - temperature gradients in the bath; INDX - thermometer to be calibrated; other symbols see Fig. 13 and text.

Table 1: Example for the evaluation of the input quantities

Quantity	Knowledge available	PDF	Expectation value	Uncertainty contribution
t_{INDX}	4 observations: mean : 19,86000 °C SD : $8{,}16 \cdot 10^{-3}$ °C	gaussian (PME)	19,86000 °C	$4{,}08 \cdot 10^{-3}$ °C
t_{INDS}	4 observations: mean : 20,00625 °C SD : $8{,}54 \cdot 10^{-3}$ °C	gaussian (PME)	20,00625 °C	$4{,}27 \cdot 10^{-3}$ °C
δt_{BathX}	manufacturer`s statement: max. deviation $\pm 15 \cdot 10^{-3}$°C	rectangular	0	$8{,}65 \cdot 10^{-3}$ °C
Δt_S	calibration certificate: error:: -0,05 °C *U(k=2)* $20 \cdot 10^{-3}$°C	gaussian (PME)	- 0,05 °C	$10{,}00 \cdot 10^{-3}$ °C

5. CONCLUSIONS

The modelling procedure presented is applicable to the most areas of uncertainty evaluation of measurements performed in the steady state. It is clearly structured into five elementary steps, and only three types of modelling components are employed. This procedure has been successfully presented in uncertainty training courses attended by a total of more than 400 engineers and physicists.

It may be utilized for an interactive computer-controlled modelling and uncertainty-evaluating procedure.

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