XVII IMEKO World Congress
Metrology in the 3rd Millennium
June 22–27, 2003, Dubrovnik, Croatia

REPRODUCING KERNEL HILBERT SPACE METHODS IN MEASUREMENT SCIENCE

Tatiana Siraya

Central Scientific Research Institute "Elektropribor", St.-Petersburg, Russia

Abstract − Basic functional model for random signal is presented for use in measurement theory. That is H(R) - reproducing kernel Hilbert space (RK-space), produced by the correlation function R(s, t) of random process x (t).

RK-space H(R) presents an isomorphic representation of the process x(t). So it provides an adequate mathematical tool for solving several problems, such as linear filtering, extrapolation of random signal, and deterministic signal extraction from noise. Besides, the corresponding RK-norms are useful in metrology for employing as the measurement accuracy characteristics.

As an illustration of the RK-approach, the pseudo-best B-estimates for the deterministic signal extraction from noise are considered.

Keywords:measurement, random process, characteristic.

1. INTRODUCTION

Random processes are commonly accepted in measurement theory and data processing as the basic models of signals. In practice signals are usually considered as stationary processes, which allow spectral representations. So the most of investigations and estimates are based on the spectral models.

However, in practice there are many important kinds of signals, which have no stationary property. So there are no classical spectral representations for such signals. The problem of non-stationary signal investigation is of great importance for the measurement theory in general, and it is of prime significance for data processing in measurements.

Several kinds of generalized or modified spectral representations have been proposed for some classes of non-stationary random processes, but each of them has a restricted scope of application. For instance, there is a class of harmonizable processes, which have the generalized spectral representation. But in this case spectral measure proves to be two-dimensional; therefore, Fourier transform lack its valuable properties, which are essential for use in stationary case. So harmonizable processes are not very useful for practice

Thus time-domain approach seems to be promising for measurement theory. Relevant time-domain models are to be investigated in order to represent the time-domain properties of signals. They form the basis for defining of the signal characteristics and for deriving the corresponding estimates.

2. METHODS AND RESULTS

2.1. Basic Model – Functional RK – Space

In this paper the time-domain approach is developed for the representation of non-stationary signals x (t), $t \in [0, T]$. Functional Hilbert space H(R), based on correlation function R(s, t), s, $t \in [0, T]$, is considered as a basic functional model [1]. This is a Hilbert space of functions on the interval [0, T] with the reproducing kernel R(s, t), or RK-space. It contains all the function of the form $R_t = R(\cdot, t)$, $t \in [0, T]$. The inner product in RK-space H(R) is defined by the condition:

$$(g, R_t)_R = g(t). \quad (1)$$

Functional space H(R) is very useful for investigation of the random process x(t) with correlation function R(s, t). In particular, it provides an isomorphic representation of the stochastic Hilbert space H(x), generated by random values x(t) according mean square norm. The isomorphism [2, 3]:

$$F: H(x) \rightarrow H(R), \quad (2)$$

converts the random value $x(t) \in H(x)$ into the function $R_t = R(\cdot, t) \in H(R)$. So a given correlation function R(s, t) just defines the specific metric in the RK-space H(R):

$$(R_s, R_t)_R = (x(s), x(t)) = R(s, t). \quad (3)$$

In measurement practice, there are several tasks concerning random signals, which are formulated as linear problems in stochastic Hilbert space H(x). In particular, these are problems of linear filtering and extrapolation, and deterministic function extraction from noise.

Owing to the isomorphism (2), the RK-space H(R) provides a ready mathematical tool for solving the mentioned problems in the well-defined functional space. The level of this tool efficiency depends on the expediency and ease of the major operations in the RK-space.

So it is important to investigate the properties of RK-spaces and the operations in these spaces. In particular, the properties of RK-spaces are to be compared with the spectral representations.

2.2. Comparison of RK-models and spectral representations

For many major classes of processes, which are topical for measurement problems, the RK-spaces may be presented in direct form.

Firstly, for the stationary processes RK-representations are directly related with the spectral ones; in some sense, these two kinds of models are quite equivalent. In particular, if x(t), t ∈(-∞, ∞), is the stationary process with spectral density f(λ), then RK-space H(R) consists of Fourier transforms of square-integrable functions with the weight f:

$$H(R) = \{g: g(s) = \int exp(-is\lambda)\, h(\lambda)\, d\lambda,\ h \in L^2(f)\ \} \qquad (4)$$

$$L^2(f) = \{h:\ \|h\|_f^2 = \int |\ h(\lambda)\ |^2 f(\lambda)d\lambda < \infty\}. \qquad (5)$$

Thus, all the "spectral" properties of stationary processes may be simply reformulated in terms of RK-space.

Apart from the stationary random processes, the direct representations are valid for the non-correlated process (white noise), Brownian motion, processes with non-correlated increments, Markovian and N- Markovian processes, and some other types of random signals.

For these types of processes the detailed structure of the corresponding RK-spaces has been studied. In particular, expressions for scalar products and norms are obtained and analyzed. Further, corresponding estimates based on experimental data are derived and studied. These estimates are the appropriate characteristics of the random signals.

For some cases RK-spaces have rather simple form. In particular, if $x_1, ..., x_n$ is non-correlated time series (or random sample), then RK-space is just l^2-vector space with the usual square norm. So the appropriate estimate is just the classical sample variance.

Likewise, Allan variance [4] corresponds to the norm in RK-space, produced by the random process with non-correlated increments, or innovation process. So Allan variance may be also called as "innovation variance" [5].

RK-representations are also valid for the generalized processes. In the case of generalized white noise process, Hilbert space H (R) is just the space of the square-integrable functions L^2 on the interval [0, T]:

$$H(R) = \{g:\ \|g\|^2 = \int |\ g(t)\ |^2\, dt < \infty\}. \qquad (6)$$

Properties of RK-representations clearly demonstrate the advantages of these models, so they are worth wider using in practice. RK-representations were introduced in 1960s [2, 3], and they were widely used in the theory of random processes [6]. An important advantage of RK-representation over the spectral one is that the former is applicable for both stationary and non-stationary processes. RK-spaces are extremely simple and useful for the white noise, Brownian motion and processes like those.

But there are naturally some imperfections in the RK-spaces. They are clearly seen in comparison with spectral representation, which have rather general and unified form. Besides, all the spectral operations are quite determined by the well-known properties of the Fourier transform.

On the other hand, RK-spaces are too individual and specific. They are so closely linked with the particular kernels R(s, t), that the intrinsic metrics in RK-spaces usually have rather special form. In particular, the basic functions $R_t = R(\cdot, t)$ may be of complicated form.

So, RK-spaces have the merits, such as individual character and direct presentation of the process values. But it turns into disadvantages, as it tends to the complicated norms and unusual metrics in RK-spaces. This is the reason that the practical employment of RK-models is still limited.

2.3. Ordering of RK-spaces

The functional RK-model has an important property that it is possible to establish a partial order of the models. Firstly, the notions of subordinated and dominated kernels are introduced in the following way. The kernel $R_2(s, t)$ dominates $R_1(s, t)$ (and R_1 is subordinated to R_2): $R_1 \le R_2$, if the difference

$$R_0(s, t) = R_2(s, t) - R_1(s, t) \qquad (7)$$

is the positively defined kernel [1, 7].

As applied to the corresponding random processes, it means that the process $x_1(t)$ with the correlation function $R_1(s, t)$ may be obtained as a projection of the process $x_2(t)$ with the correlation function $R_2(s, t)$ onto a certain subspace H_1 in the space of random values.

The ordering relation for the kernels generates the corresponding ordering relation for the RK-spaces. If the kernel $R_2(s, t)$ dominates $R_1(s, t)$, then RK-space $H(R_1)$ is the subspace of the RK-space $H(R_2)$. In this case $H(R_1)$ can be also called as subordinated to the space $H(R_2)$.

There is a practically important case, then the actual correlation function of the process $R_1(s, t)$ is not known, but it is possible to construct or define the kernel $R_2(s, t)$, which dominates $R_1(s, t)$. Thus all the principal problems mentioned above, such as linear filtering, extrapolation, and the signal extraction from noise, may be solved using the dominating RK-space $H(R_2)$, instead of the actual RK-space $H(R_1)$.

Therefore, it is practically important to reveal a relatively limited set of such "basic" processes, that the corresponding RK-spaces would be easy to describe and handle. It is also preferable, that these RK-spaces would be rather extensive; in this case they would dominate a wide range of RK-spaces, corresponding to the actual signals. So a set of RK-models should be established with the desired propertied stated.

The set of basic RK-model should certainly include stationary processes and several classes of non-stationary ones, such as non-correlated process (or white noise); processes with non-correlated increments; Markovian and N-Markovian processes. It is necessary to investigate the detailed structure of the corresponding RK-spaces.

In particular, if $x_1, ..., x_n$ is time series with non-correlated increments, the corresponding RK-space is the vector space with norm

$$\|\ x\ \|^2 = |\ x_0\,|^2 + \Sigma\,|\ x_{k+1} - x_k\ \ |^2 \qquad (8)$$

For the Brownian motion with the correlation function

$$B(s, t) = \sigma_0^2 + \sigma^2 \min(s, t), \qquad (9)$$

RK-space H (R) is just the space of the absolutely continuous functions with the square-integrable derivatives:

$$H(B) = \{f:\ \int |\,f'(u)\,|^2\, du < \infty\}. \qquad (10)$$

Likewise, the general process with non-correlated increments has the correlation function of the form

$$B(s, t) = \sigma_0^2 + \sigma^2 \ \mu(0, min(s, t)), 0<s, t<T, \qquad (11)$$

with μ (0, s) being a certain measure of the interval [0, T]. Then RK-space H (R) is just the space of the functions, which are absolutely continuous relatively measure μ, with the square-integrable derivatives:

$$H(B)=L_1^2\{\mu\}= \{f: \int(df(u)/d\mu(u))^2 d\mu(u) < \infty\} \qquad (12)$$

So the scalar product in this RK-space is the following:

$$(f_1, f_2)_B = \int(df_1(u)/d\mu(u))\ (df_2(u))/d\mu(u))\ d\mu(u) \qquad (13)$$

In the case of Markovian process the correlation function is represented in the form:

$$R(s, t) = \psi(s)\ \varphi(t), 0<s<t<T, \qquad (14)$$

where ψ (s) and φ(t) are continuous functions,
and the ratio u(t)=ψ (t)/φ(t) is an increasing function.
Then RK-space consists of the functions, which have the following representations:

$$f(t)= \varphi(t)\ g(t), \ \int(g'(t))^2 / u'(t)\ dt < \infty \qquad (15)$$

So the scalar product in this RK-space is the following:

$$(f_1, f_2)_R = \int(g_1'(t)\ g_2'(t) / u'(t)\ dt\ . \qquad (16)$$

As applied to these RK-models, the corresponding classes of signals are formed with RK-spaces subordinated to these models. It is important, that the necessary and sufficient conditions of subordination are also formulated in terms of RK-spaces.

The RK-models of Brownian motion or non-correlated increments process are extremely significant. In particular, they produce the RK-norms, which are closely related with Allan variance [4]. The latter is an important characteristic of data scatter, which is useful both for many kinds of measurements [8].

It is to be noted, that the classes of random signal, formed as subordinated to basic RK-models, do not produce a proper classification system. The classes are not disjoint, but they are complementary in some sense. So there are many processes, which are simultaneously subordinated to several RK-models.

The RK-approach to the signal representation may be called "coordinate free", but by the introduction of suitable parametric systems, it can be done parametric.

It may be also called "stationary free". Indeed, the stationary case is just one of the basic RK-models, but the class of subordinated processes includes both stationary and non-stationary ones. On the other hand, any basic RK-model mentioned above dominates some stationary processes. So this approach is independent of stationary property, and it provides an essential linking between stationary and non-stationary cases.

2.4. Signal extraction from noise

As an illustration of the RK-approach, the problem of extraction of deterministic signal from noise is considered. It is very important for measurement practice; in particular, it may be used for estimation of the systematic errors of the measuring device.

The signal under observation is supposed to be a sum of given functions f_1, ..., f_m with unknown weights $a_1, ..., a_m$,:

$$f(t) = \Sigma\, a_i f_i(t) \qquad (17)$$

It is observed in the discrete points within the interval [0, T], but it is corrupted by random noise x(t):

$$y_k = y(t_k) = \Sigma\, a_i f_i(t_k) + x(t_k), \ 0 \le t_k \le T\ . \qquad (18)$$

Correlation function R(s, t) of the noise x (t) is usually unknown. But often it is possible to find a correlation function B(s, t), that dominates R(s, t).

In this case one cannot construct the optimal linear estimates of parameters $a_1, ..., a_m$; but it is possible to find the pseudo-best B-estimates, assuming correlation function B(s, t) instead of R(s, t) [7]. These estimates are constructed like the classic least squares estimates, using RK-space H(B) norm (instead the sum of squares). So the system of equations for finding parameters $a_1, ..., a_m$, is just as follows:

$$\Sigma\,(f_i, f_j)_B\, a_j = (y, f_i)_B\ , i = 1...m. \qquad (19)$$

B-estimates are the direct generalization of the classic least squares estimates; the latter correspond to the white noise RK-space H(B). Apart from this case, B-estimates formed according to Brownian motion, and Markov processes are of practical interest.

Certainly, B-estimates are not optimal ones. For the practical application of B-estimates it is necessary to study the basic properties of these estimates, in particular, obtain the conditions for the following:

1) statistical consistency of B-estimates, when they converge to the true values of parameters;
2) asymptotical efficiency of B-estimates, when they are almost as accurate as the optimal estimates;
3) sufficient relative efficiency of B-estimates, when there is not so much loss of accuracy caused by using B-estimates instead of the optimal one.

All the conditions and estimates mentioned above can be formulated in terms of the corresponding RK-spaces. They depend on the interrelations of the spaces H(B) and H(R), and also of the properties of the functions f_1, ..., f_m. These conditions may be given in the explicit form for the cases of Brownian motion, and Markov kernel B.

The general condition of consistency is equivalent that RK-space H(R) is just a subspace of H(B), or the kernel R is subordinated to B.

In particular, it can be directly formulated for the case of Brownian motion kernel B.

a) Stationary kernel with spectral density f(λ) is subordinated to Brownian motion, if and only if the function { λ^2 f(λ) } is bounded.
b) Stationary increment kernel with spectral density f(λ) is subordinated to Brownian motion, if and only if the spectral function f(λ) is bounded.
c) Markovian correlation function of the form (14) is subordinated to Brownian motion, if the functions answer requirements:

$$\int(\psi'(t))^2 / u(t)\ dt < \infty$$

$(\psi(t))^2 \; u'(t)$ is bounded.

The study of B-estimates shows that B-estimates are rather simple and convenient. They also give an acceptable accuracy in practice.

3. CONCLUSIONS

The time–domain approach and functional RK-models, studied in this report, forms a methodological basis for development of the signal processing methods, which are useful for metrological practice.

RK- approach extends the functional representations for non-stationary processes. It provides the opportunities for solving problems of linear filtering, and extrapolation of non-stationary signals. This approach also gives some new methods for the statistical signal processing, such as pseudo-best B-estimates. Besides, this approach is useful for introducing the measurement accuracy characteristics

RK-approach also gives a new insight into the relationship between the well-known statistical models with corresponding characteristics, and more general non-stationary models with other kinds of characteristics.

REFERENCES

[1] Aronszajn N. Theory of reproducing kernels. – Trans. Amer. Math. Soc., 68 (3), 1950.

[2] Hajek J. On statistical problems in stochastic processes. – Czechoslovakian Mathematical Journal, 12 (87), 1962.

[3] Parzen E. A new approach to the synthesis of optimal smoothing and prediction systems.– In “Mathematical optimization techniques”, Univ. California Press, 1963.

[4] Allan D.W., Ashby N., Hodge C.C. The Science of Timekeeping. – Application Note 1289, Hewlett-Packard Company, 1997.

[5] Siraya T.N. Comparison of Uncertainty Estimates: Allan Variance and Sample Variance. – In: “Proceedings of 3rd International Conference on Measurement “Measurement-2001”, Bratislava, SAV, 2001.

[6] Kallianpur G. The role of Reproducing kernels Hilbert space in the study of Gaussian processes. – In “Advances in probability and related topics”, N. Y., 1971.

[7] Tempelman A. A. On linear regression estimates. – In “2nd International Symposium on information theory” – Budapest, 1973.

[8] Witt T. J. Testing for correlations in measurements – In “Advanced Mathematical and Computational Tools in Metrology,” IV, 2000.

Author: D. Sc. Tatiana N. Siraya, Leading researcher,
Central Scientific Research Institute “Elektropribor”,
197046, St.-Petersburg, Russia.
Fax: 7 (812) 232 33 76. E-mail: elprib@online.ru