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RESEARCH OF THE CHANGE TREND OF EVALUATION ERRORS OF THE COVERAGE FACTOR IN INDIRECT MEASUREMENTS

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Abstract − The results of research of evaluation error of the coverage factor with approximate method of effective number degrees of freedom in indirect measurements have been presented in the paper. Comparison of the results obtained with the known evaluation of these errors in direct measurements enabled to determine the change tendency of the errors of coverage factor evaluation, when the number of components of standard uncertainties grows. The knowledge of coverage factor characteristics for the convolution of three selected probability distributions was used for the research.

Keywords uncertainty, coverage factor

1. INTRODUCTION

Each evaluation of the expanded uncertainty requires the choice of an approximate evaluation method of the coverage factor. In the methods suggested by the international document [1] it is necessary to decide whether the evaluated factor shall approach the factor for a normal distribution or for Student's distribution. Usually the sample size is the decisive factor in the choice. However, how the number of standard component uncertainties influences the choice of the evaluation method is unknown. When deciding about the method of expanded uncertainty estimation, one should be aware of the effects of choosing a particular method from the viewpoint of its accuracy. The basis for estimating the accuracy of applied approximate methods of the estimation of expanded uncertainty is the assumption on the necessity the assessment methods, which could be regarded as exact estimation.

An essentially appropriate concept was adopted, which is taken into consideration, that the method based on the command of the convolution of component distributions may be regarded as an exact method. Due to complexity and time-consuming character of computing the convolution of many distributions of components, the results of such computing are, in general, hardly ever published. Therefore, approximate methods are generally accepted and recommended.

There are the results of publications [2], [3], [4], [5], concerning the analysis of accuracy of approximate methods of expanded uncertainty estimation for simple direct measurement, when there are only two component standard uncertainties.

In the present paper the analysis of accuracy of estimating the coverage factor in indirect measurements was described.

Description of the convolution of two Student's distributions and one rectangular distributions S*S*R is presented hereafter as well as the calculation results of the selected values of the coverage factor for the convolution. Furthermore, the factor characteristics are presented and compared to the characteristics of the coverage factor for the distribution, which is a convolution of one Student's distribution and one rectangular distribution S*R and to the characteristics of the factor $k_N(\alpha)$ for a normal distribution.

The results are presented for one selected probability value α = 0.95 and for small samples.

2. CHARACTERISTICS OF THE COVERAGE FACTORS

A measuring event, which utilizes a convolution of two Student distributions and one rectangular distribution is an example of indirect measurement carried out by means of two measuring devices, which, in case of repeated measurements, show a scatter of results, a type-B standard uncertainty of one of the devices can be neglected, and the number of measurements is small (n<30). Therefore, three standard uncertainties are analyzed: type-B standard uncertainty, which reflects a standard deviation of rectangular distribution and two type-A standard uncertainties, which reflect a standard deviation of Student distribution.

On the basis of the developed analytical description of coverage factors in case of the analyzed convolutions one is able to identify all parameters, which function are the factors. One is able to demonstrate that a coverage factor for the convolution S*S*R, from now on referred to as factor $k_{S_1S_2R}(\alpha)$ is a function of 5 variables [6]: probability α, number of degrees of freedom $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$ first and second Student's distributions and the ratio of standard uncertainties η_S and η (1):

$$k_{S_1S_2R}(\alpha) = f(\alpha, m_1, m_2, \eta_S, \eta) \quad (1)$$

where:

$\eta_S = \frac{u_{A_1}}{u_{A_2}}$ - is the ratio of standard uncertainties of type A

$\eta = \frac{u_A}{u_B} = \frac{\sqrt{u_{A_1}^2 + u_{A_2}^2}}{u_B}$ - is the ratio of combined standard uncertainty of type A to standard uncertainty of type B.

Calculations were executed for one probability value $\alpha = 0.95$, for small values m and for the value series η, ranging from 0.1 to 10.

Matlab program was used for the calculations and the following were assumed:

- approximation accuracy of the probability range α over the variable k, ε =1e-4,
- the number of integration ranges in the Simpson's method of integration 300
- multiple j =20

Computational results are presented in Table 1.

TABLE I. Values of the coverage factor $k_{S_1S_2R}(\alpha)$

1/η	η	m₁=3	m₁=9
		m₂=3	m₂=9
10		1,6754	1,6561
5		1,7650	1,7023
4		1,8211	1,7313
3		1,9184	1,7797
2		2,1313	1,8761
1	1	2,6195	2,0617
	2	3,0109	2,1844
	3	3,1297	2,2175
	4	3,1781	2,2306
	5	3,1988	2,2375
	10	3,2324	2,2453

Characteristics of the coverage factor are presented in the function of the ratio of standard uncertainties $\eta = u_A/u_B$ and its converse.

Characteristics of the coverage factor $k_{S_1S_2R}(\alpha)$ are compared to the characteristics of the coverage factor $k_{SR}(\alpha)$ for the convolution S*R and to the value of the coverage factor $k_N(0.95)$ for a normal distribution.

Fig.1 shows the characteristics of the coverage factor $k_{S_1S_2R}(0.95)$ for $m_1 = m_2 = 3$, $m_1 = m_2 = 9$, $\eta_S = 1$ in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse.

Broken line shows characteristics of the coverage factor $k_{SR}(0.95)$ for the convolution S*R, for $m = 3$ and $m = 9$, and the coverage factor $k_N(0.95)$ for a normal distribution.

In this situation both samples have the same number of degrees of freedom, and none of the component standard uncertainties of type A is a domineering one.

In accordance with the central limit theorem, the characteristics of the coverage factor $k_{S_1S_2R}(0.95)$ and $k_{SR}(0.95)$ clearly trend to approach the value of the factor $k_N(0.95)$ as the sample size increases. The phenomenon is observed in the domain where $u_A > u_B$, further called domain A.

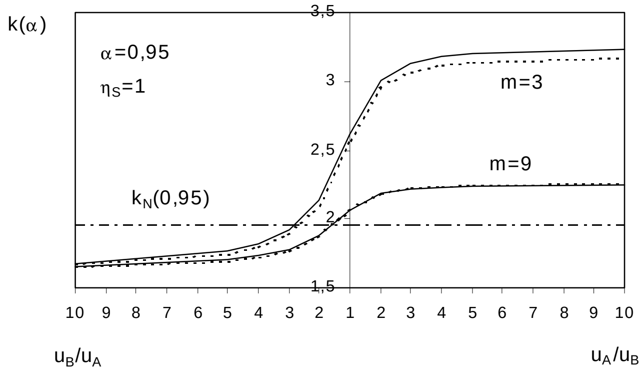

Fig. 1 Characteristics of the coverage factor $k_{S_1S_2R}(0.95)$, $k_N(0.95)$ and $k_{SR}(0.95)$ in the function η and its converse

Whereas in the domain where $u_B > u_A$, further called domain B, the influence of the sample sizes is much smaller and fades as the values of the ratio $u_B > u_A$ increases.

3. THE METHOD OF EFFECTIVE NUMBER DEGREES OF FREEDOM

Document [1] suggests for a measuring event with a small number of tests a method according to which the coverage factor $k(\alpha)$ assumes the values of standardized variable of Student distribution $k_{m_e}(\alpha)$, read from the table of this distribution for the effective number degrees of freedom m_e.

According to Welch-Satterthwaite'a Formula [3], if the combined standard uncertainty is a root of a sum of two or more variances estimated on the basis of results of not numerous test with unknown standard deviation σ, the unknown distribution of the required standardized variable can be approximated by means of a Student distribution for the effective number degrees of freedom m_e.

In the considered case of indirect measurement the effective number of degrees of freedom is described by means of relationship (2) resulting from the general Welch-Satterthwaite'a formula:

$$m_e = \frac{u_{\bar{c}}^4}{\sum_{j=1}^{N} \frac{1}{m_{A_j}} \left(\frac{\partial f}{\partial x_j} \right)^4 \cdot u_{A_j}^4 + \sum_{j=1}^{N} \frac{1}{m_{B_j}} \left(\frac{\partial f}{\partial x_j} \right)^4 \cdot u_{B_j}^4} \qquad (2)$$

where:

- $u_{\bar{c}}$ is a standard combined uncertainty of value Y measured indirectly and computed according to the uncertainty propagation law
- $m_{A_j} = n_j - 1$ is the number of degrees of freedom of the j -th measurement

- m_{B_j} is the number of degrees of type B freedom and is computed on the basis of reliability of component standard uncertainty of type B.

In a situation when type-B standard uncertainty is estimated on the basis of known rectangular distribution which borders are defined by the limiting error of measuring devices, one can assume that this uncertainty is well known. Therefore, for the following analysis one can assume that the relative uncertainty of type B values equal to 0.1 [2], which reflects the number of degrees of freedom m_B =50. Assuming that all partial derivatives are equal to one, and after all appropriate transformations for the analyzed situation one obtains (3):

$$m_e = \frac{\left(\eta^2+1\right)^2}{\frac{\eta^4}{\left(\eta_S^4+1\right)^2}\left(\frac{1}{m_1}\cdot\eta_S^4+\frac{1}{m_2}\right)+\frac{1}{m_B}} \tag{3}$$

This form of relationship describing m_e permits to present the characteristics of coverage factor of Student distribution for the effective number of degrees of freedom $k_{m_eSSR}(\alpha)$ in function η for various values m, in order to compare it with the characteristics of coverage factor for the analyzed convolution $k_{S_1S_2R}(\alpha)$.

Fig. 2 presents the characteristics of coverage factor $k_{m_eSSR}(0.95)$, $k_{S_1S_2R}(0.95)$ for $m_1 = m_2 = 3$ as well as for $m_1 = m_2 = 9$, $\alpha = 0.95$ and η_S =1.

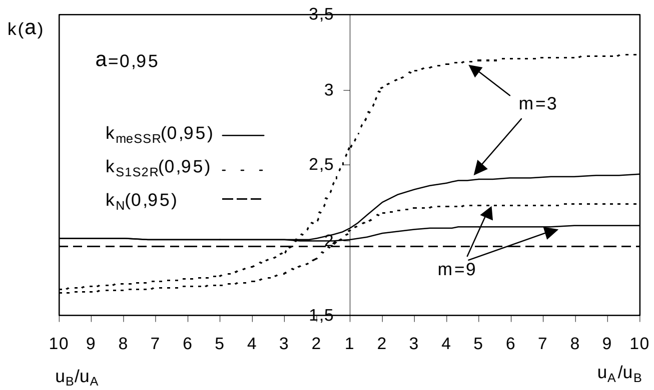

Fig. 2 Characteristics of the coverage factor $k_{m_eSSR}(0.95)$, $k_{S_1S_2R}(0.95)$ in the function η and its converse

The characteristic feature of the computed factor $k_{m_eSSR}(\alpha)$ is such that in domain A its values differ considerably from the value of factor $k_{S_1S_2R}(\alpha)$, which are exact values. With the increasing number of degrees of freedom m, the differences diminish.

In domain B the factor assumes constant values independent from the number of degrees of freedom m and the values are close to the values of the factor $k_N(0.95)$ for a normal distribution.

On the presumption, that knowledge of convolution of component distributions permits to estimate expanded uncertainty with strict accuracy it is assumed, that error, of which absolute values is described by relationship below will be measure of discrepancy between approximate and exact method:

$$\delta = \frac{|u-u_e|}{u_e}\cdot 100\% \tag{4}$$

where u - it is expanded uncertainty evaluated by means of approximate method

$$u = k(\alpha)\cdot u_c \tag{5}$$

u_e - it is expanded uncertainty estimated “exactly”, on the basis of knowledge of distribution, which in the measuring event is the convolution of two Student distributions and one rectangular distribution. The coverage factor $k_{S_1S_2R}(\alpha)$ could be regarded as exact value:

$$u_e = k_{S_1S_2R}(\alpha)\cdot u_c \tag{6}$$

In the most of considered measuring events error described by the dependence (4), will be error the estimate of unknown the coverage factor value $k(\alpha)$, which assumes form [7]:

$$\delta = \frac{|k(\alpha)-k_{S_1S_2R}(\alpha)|}{k_{S_1S_2R}(\alpha)}\cdot 100\% \tag{7}$$

According to the assumption that the value of coverage factor for the analysed convolution of component distributions may be regarded as an exact value, the absolute value of estimation error δ by means of this approximate method is defined as (8):

$$\delta = \frac{|k_{m_eSSR}(0.95)-k_{S_1S_2R}(0.95)|}{k_{S_1S_2R}(0.95)}\cdot 100\% \tag{8}$$

Fig. 3 presents the absolute error values δ of factor estimations $k_{m_eSSR}(0.95)$ in the function η for various values of m.

4. CONCLUSIONS

The influence of the sample size is the domineering influence on the error δ value in domain A. In domain B the influence of a sample size is slight and fades as the values u_B/u_A increase.

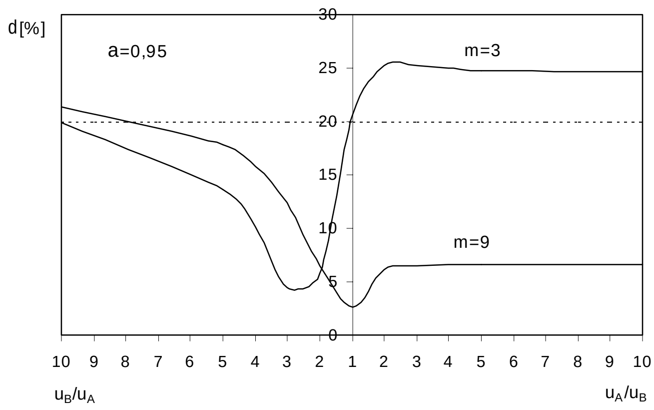

Fig. 3 Absolute error values δ of coverage factor estimations $k_{m_e SSR}(0.95)$ in the function η and its converse

According to Fig. 3 there is a limitation of applying the method of effective number of degrees of freedom for not numerous population – m=3 in a situation when we are in a domain type A. With the increasing m, the value of error decreases considerably. The results of research, which are presented in the present paper indicate only the trend of changes of errors of coverage factor estimation, according to the recommended by the international document [1] approximated method of assessment. In spite of the fact that this is the method recommended by the international document for estimating the coverage factor in indirect measurements, when short series are available, this method has limitations from the viewpoint of its accuracy.

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