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A UNIFIED CONCEPTUAL FRAMEWORK FOR SPECTRUM ANALYSER MEASUREMENTS

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Abstract - The spectrum analyser is a complex piece of instrumentation, which requires careful operator training to fully exploit its features. Over the years its structure has evolved, both as a result of technological advances and in response to new measurement requirements. A modern spectrum analyser thus offers the user an extensive array of functions and options, arranged in complex menu structures. A side-effect of this situation is that sometimes it may be difficult to retain a common conceptual view of the instrument operation, since a variety of factors and set-up parameters come into play. This paper proposes a theoretical analysis of spectrum analyser operation, with the aim of providing a conceptual framework that can be readily adapted to most practical situations.

Keywords: spectral analysis, power spectral density.

1. BASIC MODEL

Let *x(t)* be a real-valued signal having limited bandwidth *B*. The most important operation in a spectrum analyser is super-heterodyning, which is carried out in accordance with the functional diagram given in Fig. 1. The local oscillator output is assumed to be a cosine wave whose frequency f_{LO} varies linearly with time. The resulting sweep allows the analysis of the desired frequency span.

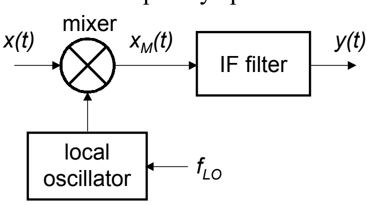

Fig. 1: supeheterodyning in a scanning spectrum analyser.

The intermediate frequency (IF) filter, centred at the frequency f_{IF}, is a symmetric passband filter, whose impulse response can be written as:

$$h(t) = w(t)\cos(2\pi f_{IF} t), \quad (1)$$

where *w(t)* is a real function called the filter envelope. It is assumed that the filter bandwidth B_H is very narrow, so that it satisfies the condition $B_H << f_{IF}$. It is known from filter theory that *w(t)* can be interpreted as the impulse response of an equivalent baseband filter, whose bandwidth is $B_W = B_H/2$. The relationships among the frequency parameters involved in superheterodyning are summarised in Fig. 2.

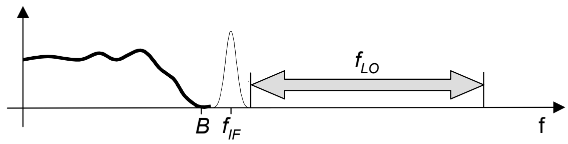

Fig. 2: frequency relationships in superheterodyning.

The output of the intermediate frequency filter is expressed by the convolution integral:

$$y(t) = \int_{-\infty}^{+\infty} K_M x(\tau)\cos 2\pi f_{LO}\tau \left[w(t-\tau)\cos 2\pi f_{IF}(t-\tau)\right] d\tau, \quad (2)$$

where K_M is the mixer conversion factor which, for the sake of simplicity, is assumed to be constant and independent of frequency.

The mixer output $x_M(t) = K_M \cdot x(t)\cos 2\pi f_{LO} t$ can be decomposed into frequency components centred around the frequencies $f_{LO} - f_{IF}$ and $f_{LO} + f_{IF}$; however, from Fig. 2 it is clear that only the former will give rise to a response at the filter output. Since the amplitude calibration factor of the instrument will account for the effects of mixing, in the following it will be assumed that $K_M = 2$. Then, by using well-known trigonometric relationships, (2) can be rewritten as:

$$y(t) = \cos 2\pi f_{IF} t \left[\int_{-\infty}^{+\infty} x(\tau)\cos 2\pi (f_{IF} - f_{LO})\tau \; w(t-\tau) d\tau\right] +$$
$$+ \operatorname{sen} 2\pi f_{IF} t \left[\int_{-\infty}^{+\infty} x(\tau)\operatorname{sen} 2\pi (f_{IF} - f_{LO})\tau \; w(t-\tau) d\tau\right]. \quad (3)$$

As the signal *y(t)* is narrowband, it can be described by a complex representation. For this purpose we introduce the function:

$$s_{f_{IF} - f_{LO}}(t) = \int_{-\infty}^{+\infty} x(\tau) e^{-j2\pi (f_{IF} - f_{LO})\tau} \cdot w(t-\tau) d\tau, \quad (4)$$

using which the IF filter output can be written as:

$$y(t) = \mathrm{Re}\left[s_{f_{IF} - f_{LO}}(t) e^{j2\pi f_{IF} t}\right]. \quad (5)$$

The function defined by (4) is called the complex envelope of *y(t)*: its magnitude is the conventional envelope of the signal *y(t)*, and the angle is the phase of the same signal with respect to the IF filter centre frequency f_{IF}. Using (4), the relationship between the input signal and the IF filter output can be expressed more concisely. In fact, equation (3) can be rewritten in the form:

$$y(t) = \left| s_{f_{IF}-f_{LO}}(t) \right| \cos\left(2\pi f_{IF} t + \arg\left[s_{f_{IF}-f_{LO}}(t) \right]\right). \qquad (6)$$

This shows that, in the general case, the response of the IF filter is an amplitude- and phase-modulated sinewave.

It is interesting to observe that (4) can also be interpreted as the Fourier transform of the function $x(\tau)w(t-\tau)$, calculated at the frequency $f = f_{IF}-f_{LO}$. In this way the filter envelope $w(t)$ is seen as a multiplying term and, by applying Fourier transform properties, the complex envelope of $y(t)$ can be expressed as a function of $X(f)$ and $W(f)$ as:

$$s_{f_{IF}-f_{LO}}(t) = X(f) * \left[W(-f) e^{j2\pi f t} \right] \Big|_{f = f_{IF}-f_{LO}}, \qquad (7)$$

where the symbol * indicates the convolution operation. Equation (7) provides an alternative way to express the result of the superheterodyning operation, but it also has an important conceptual meaning, as will be discussed in Section 5.

Depending on the architecture of the specific spectrum analyser, the IF filter output $y(t)$ can be processed in a variety of ways in subsequent instrument stages [1]. In a traditional analogue scanning analyser the envelope of $y(t)$ is recovered by means of a linear detector; further analogue stages may follow, until the resulting trace is eventually digitised. In more recent instruments, the IF filter output is digitised directly and processed by digital algorithms [2], [3].

Spectrum analysers have been provided with an ever-widening array of detection functions, designed to help the user analyse a broader variety of signals. Peak detectors, RMS detectors and FFT-based processing have all contributed to a significant improvement of the measurement capabilities of the instrument. However, somehow this forces users to think of the spectrum analyser as a set of different instruments, each one being specialised in dealing with some signal classes and having specific calibration parameters [4]. Although this can be managed after some training, from a conceptual point of view it would be desirable to provide a unified approach.

The purpose of the work presented in this paper will be to show how a common theoretical framework can provide an analytical description of spectrum analyser measurements that encompasses most cases of practical interest, regardless of the specific instrument architecture.

2. POWER ANALYSIS

The discussion will be restricted to scalar analysers, therefore only the amplitude information contained in $y(t)$ will be considered in the following. Usually, this means that a linear detector is employed to obtain the envelope of the IF filter output, $\left| s_{f_{IF}-f_{LO}}(t) \right|$. However, spectrum analyser measurements are more commonly considered in terms of power, thus it becomes more natural to think of the corresponding power function as the object of the analysis. At this point it is important to recall that the complex envelope depends on two variable quantities: time and the local oscillator frequency f_{LO}. With a correct instrument set-up the frequency sweep is slow enough that a quasi-stationary behaviour can be assumed for the latter. Therefore, the measured trace can be seen as a succession of constant values of f_{LO}, making the analysis simpler. It is also useful to introduce a change of variables, which follows naturally by noting that the abscissa on the spectrum analyser display is calibrated to indicate directly the analysed signal frequency f_x. This quantity is obtained simply as $f_x = f_{LO}-f_{IF}$. Accordingly, the trace $T(f_x, t)$ that is considered in the following is defined by the expression:

$$T(f_x, t) = \left| s_{-f_x}(t) \right|^2 = s_{-f_x}(t) \cdot s^*_{-f_x}(t). \qquad (8)$$

In the simple case of a sinusoidal signal, $x(t) = A_0 \mathrm{sen}(2\pi f_0 t + \varphi_0)$, the complex envelope can be obtained from (7). Considering the range of variation of f_{LO}, as shown in Fig. 2, and assuming the IF filter selectivity is enough to have $|W(f)| \cong 0$ for $|f| > f_0$, only one of the two frequency components of the sinewave will be present in the spectrum analyser trace, yielding:

$$T(f_x, t) = \frac{A_0^2}{4} \left| W(f_x - f_0) \right|^2. \qquad (9)$$

In this case there is no dependence on t, which means the displayed trace will not change with time. As can be expected, the trace will peak at $f_x = f_0$, with an amplitude proportional to half the signal power.

3. RANDOM PROCESSES

For analysis purposes, several signals (e.g., digitally modulated telecommunications carriers) are better represented as random processes. It will now be assumed that the analysed signal $x(t)$ be a zero-mean stationary random process having power spectral density $S_{xx}(f)$.

When the IF filter bandwidth B_H is narrow in comparison to the signal bandwidth B (i.e., $B_H << B$), the power spectral density (PSD) of $x(t)$ is approximately constant within the IF filter passband. Therefore, in that interval one has $S_{xx}(f) \cong S_{xx}(f_{IF} - f_{LO})$. Introducing again the change of variables $f_x = f_{LO}-f_{IF}$, and recalling that $S_{xx}(f)$ is even symmetric, the PSD of the random process $y(t)$, that represents the IF filter output, is expressed by:

$$S_{yy}(f) = S_{xx}(f_x) \cdot \tfrac{1}{4} \left[\left| W(f - f_{IF}) \right|^2 + \left| W(f + f_{IF}) \right|^2 \right]. \qquad (10)$$

In view of its consequences, the importance of the assumption that $B_H << B$ should be emphasised. In fact, the duration of the IF filter impulse response is proportional to $1/B_H$, whereas the correlation of $x(t)$ tends to vanish when delay values get larger than $1/B$, that is, in a much shorter time. It follows that $y(t)$ can be interpreted as the linear combination, through $h(t)$, of a large number of uncorrelated realisations of $x(t)$. The central limit theorem, then, implies that the IF filter output is a gaussian process.

At this stage, it would be possible to follow the same approach introduced in Section 1. However, further analysis is complicated by the fact that (4) would now represent a complex random process with correlated real and imaginary parts. It is more useful to consider instead an equivalent process having the same second-order description, but better analytical tractability. With this aim, the narrow band gaussian process *y(t)* can be replaced by the weakly equivalent random process [5]:

$$z(t) = \cos 2\pi f_{IF} t[u(t)] + \text{sen}\, 2\pi f_{IF} t[v(t)], \quad (11)$$

where f_{IF} is the IF filter centre frequency; *u(t)* and *v(t)* are uncorrelated, zero-mean, stationary and gaussian processes. The PSD's of the two baseband processes are:

$$S_{uu}(f) = S_{vv}(f) = S_{xx}(f_x) \cdot \tfrac{1}{2}|W(f)|^2, \quad (12)$$

which implies that *z(t)* has the same PSD as *y(t)*. Then, the (weakly equivalent) complex envelope becomes $s_{-f_x}(t) = u(t) - jv(t)$, from which the mathematical expression of the spectrum analyser trace can be obtained:

$$T(f_x,t) = s_{-f_x}(t) \cdot s^*_{-f_x}(t) = u^2(t) + v^2(t) \quad (13)$$

When the analysed signal is described by a random process, the resulting spectrum analyser trace will be affected by random fluctuations. Therefore it is necessary to consider its expected value which, from (12) and (13), is:

$$E[T(f_x,t)] = S_{xx}(f_x) \cdot \int_{-\infty}^{+\infty} |W(f)|^2 df . \quad (14)$$

It should be noticed that the expectation is conditional on the local oscillator frequency taking the fixed value f_{LO}.

Equation (14) shows that the mean value of the trace provides an indication that is proportional to the PSD of the analysed random process. In practice, it has to be assumed that a number of traces will be acquired and their average $\overline{T}(f_x,t)$ is taken as the measurement result.

It is interesting to compare the integrals of traces (9) and (14) for positive values of f_x. For a sinewave, recalling the assumption that $|W(f)| \cong 0$ when $|f| > f_0$, one has:

$$\int_0^{+\infty} T(f_x,t) df_x = \frac{A_0^2}{4} \int_0^{+\infty} |W(f_x - f_0)|^2 df_x = \frac{A_0^2}{4} \int_{-\infty}^{+\infty} |W(f)|^2 df . \quad (15)$$

In the case of a random process, the integral for positive frequencies yields:

$$E\left[\int_0^{+\infty} T(f_x,t) df_x\right] = \int_0^{+\infty} S_{xx}(f_x) df_x \cdot \int_{-\infty}^{+\infty} |W(f)|^2 df =$$
$$= \frac{\sigma_x^2}{2} \cdot \int_{-\infty}^{+\infty} |W(f)|^2 df . \quad (16)$$

In both cases, the result is equal to half the signal power, as would be expected by considering only positive frequencies in a two-sided Fourier transform. It is apparent, therefore, that the trace defined by (8) may allow to present the results of spectral analysis by a scanning superheterodyning instrument in a unified form. This claim has to be checked against yet one more case of interest.

4. PULSE SPECTRA

The case commonly referred to as a pulse spectrum occurs when the signal spectrum is composed of a large number of spectral lines, narrowly spaced along the frequency axis and having a more slowly varying amplitude envelope. The spectrum can be expressed as:

$$X(f) = \sum_{k=-\infty}^{+\infty} G(kF)\delta(f - kF), \quad (17)$$

where F is the (uniform) spacing between adjacent lines. In this situation it may be difficult to find a satisfactory instrument set-up for measuring the line spectrum. Since the amplitude envelope may have greater interest, the analyser is used to measure it as a continuous function of frequency.

Since the IF filter bandwidth is wider than the spectral line separation, several components may fall within the passband simultaneously. Nevertheless it is still possible to determine the complex envelope of *y(t)* from (7):

$$s_{-f_x}(t) = \left. \sum_{k=-\infty}^{+\infty} G(kF)W(-f - kF) \cdot e^{j2\pi(f-KF)t} \right|_{f=-f_x} . \quad (18)$$

It can be seen that the trace (8) computed from this expression will be composed of two terms, one being constant, while the other varies with time and is due to cross-terms in the multiplication of $s_{-f_x}(t)$ by its complex conjugate.

As a simple example, let the analysed signal be composed of just two sinewaves:

$$x(t) = A_0\, \text{sen}(2\pi f_0 t + \varphi_0) + A_1\, \text{sen}(2\pi f_1 t + \varphi_1), \quad (19)$$

where $f_0 < f_1$ and the difference $f_1 - f_0$ be smaller than the IF filter bandwidth B_H. The resulting complex envelope is the sum of two terms:

$$s_{-f_x}(t) = j\frac{A_0}{2} W(f_x - f_0) e^{j[2\pi(f_x - f_0)t - \varphi_0]} + $$
$$+ j\frac{A_1}{2} W(f_x - f_1) e^{j[2\pi(f_x - f_1)t - \varphi_1]} . \quad (20)$$

A graphical representation is given by the vector diagram of Fig. 3.

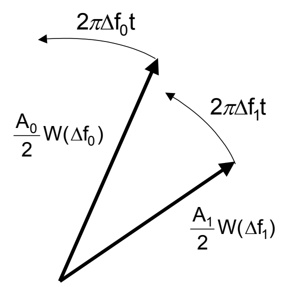

Fig. 3: vector diagram showing the composition of terms in (20).

It is important to notice that the two vectors have different angular velocities $\Delta f_0 = f_x - f_0$ and $\Delta f_1 = f_x - f_1$. Consequently, for any value of f_x, the trace computed from (20) contains a constant term, as well as a sinusoidal term having frequency $f_1 - f_0$. The latter cannot be eliminated by the envelope detector, whose bandwidth must be wider than B_H whereas, by assumption, $f_1 - f_0 < B_H$. However, it is possible to get rid of this term, for instance, by averaging the trace, since the phase of the sinusoidal term varies randomly from one sweep to the next. The situation is summarised in Fig. 4 for the case of equal amplitude components. The continuous line shows the trace obtained after averaging, while dotted lines have been used to indicate both the appearance of a single trace and the position of the two components.

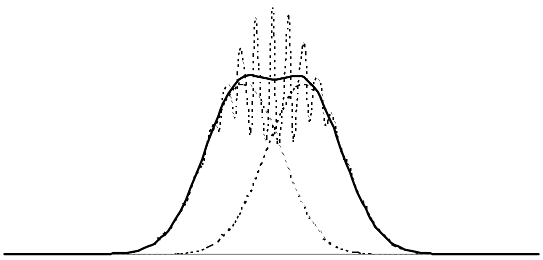

Fig. 4: spectrum analyser trace for two closely spaced lines.

In the general case, then, the averaged trace $\overline{T}(f_x,t)$ yields:

$$\overline{T}(f_x,t) = \sum_{k=-\infty}^{+\infty} |G(kF)|^2 |W(f_x - kF)|^2 . \qquad (21)$$

Considering once more the integral for positive frequencies one has:

$$\int_0^{+\infty} \overline{T}(f_x,t) df_x = \sum_{k=0}^{+\infty} |G(kF)|^2 \cdot \int_{-\infty}^{+\infty} |W(f)|^2 df \qquad (22)$$

which shows that, with the added assumption of post-detection filtering, the trace integral is still proportional to half the signal power.

5. UNIFIED FRAMEWORK

The analysis presented in the previous sections has evidenced that it is always possible to think of the trace in a scanning spectrum analyser as a measurement of the power spectral density of the analysed signal. Remarkably, this holds true also for signals that are not represented by random processes. As a logical consequence, it should be possible to deal with spectrum analyser measurements in a unified way, regardless of the analysed signal class.

It should be remembered that (8) is a theoretical definition, that does not always correspond with the actual trace on the analyser screen. As recalled in Section 1, the envelope detector output can be further processed and, usually, different calibrations are possible for the vertical (power or amplitude) scale of the spectrum analyser. In particular, the traditional approach is to calibrate the scale so that, given a sinewave input, the peak value of (9) directly corresponds to its power (or rms value). In this case the calibration factor would be $2/[|W(0)|^2 \cdot |Z_{in}|]$, where Z_{in} is the spectrum analyser input impedance. By comparison, (14) requires an entirely different factor to provide a PSD measurement. In fact, (14) can be rearranged in the form:

$$E[T(f_x,t)] = S_{xx}(f_x) \cdot ENBW \cdot |W(0)|^2 , \qquad (23)$$

where $ENBW$ is the equivalent noise bandwidth of the IF filter, defined as:

$$ENBW = \frac{\int_{-\infty}^{+\infty} |W(f)|^2 df}{|W(0)|^2} . \qquad (24)$$

The required calibration factor is $1/[ENBW \cdot |W(0)|^2 \cdot |Z_{in}|]$, or, better, $2/[ENBW \cdot |W(0)|^2 \cdot |Z_{in}|]$ if the trace is to be interpreted as a one-sided (positive frequencies only) PSD. Of course the "PSD calibration factor" differs both in value and in meaning from the "sinewave power calibration factor", making the two measurements incompatible. This problem is addressed in some instruments by providing a dedicated "noise marker" [6]. On the other hand, (15) shows that sinewave power measurements can be obtained also when the "PSD calibration factor" is employed, provided the trace is suitably processed. It follows that a unified interpretation of spectrum analyser measurements can be obtained if the "PSD calibration factor" is applied for all signal classes.

The considerable similarity of results given by (15), (16) and (22) can be traced back to the frequency domain interpretation of the complex envelope, given by (7). It is apparent that the filter envelope *w(t)* can be thought of as a windowing function for the analysed signal; in fact, in any practical filter realisation *w(t)* has finite duration, effectively giving a finite observation time for any frequency component of the signal *x(t)*. Thus, even though the operating principle is entirely different, the behaviour of a scanning spectrum analyser can be directly compared to that of Fourier-based digital spectrum analysers. Likewise, (9) shows the effect of what, in Fourier analysis parlance, would be termed spectral leakage, with the frequency response of the IF filter baseband equivalent, *W(f)*, taking the role of the window spectrum. The resolution bandwidth B_R is the -3 dB bandwidth of *W(f)*, therefore $B_R = B_H$.

It should be noticed that the value of $ENBW$ defined in (24) differs from that of the resolution bandwidth B_R; however, the ratio $ENBW/B_R$ is a filter design parameter and can be considered constant within a given instrument, whereas B_R can be changed according to user needs. Thus, the one-sided PSD calibration factor can be rewritten in the final form:

$$K_{PSD} = \frac{1}{B_R} \cdot \frac{2}{\frac{ENBW}{B_R} \cdot |W(0)|^2 \cdot |Z_{in}|} , \qquad (25)$$

where only the term $1/B_R$ needs to be changed whenever the user-selected resolution bandwidth is changed.

6. MEASUREMENT WITH SAMPLED TRACES

The spectrum analyser usually displays a sampled trace. Therefore, all related expressions should actually be considered in sampled form with the number of samples available, N, being finite. Thus, within the selected frequency span F_{SPAN} the spectrum will be measured only at the frequencies $f_x = kF$, with k being an integer and $F = F_{SPAN}/N$. However, sampling can be carried out in a number of different ways; often, the sampling circuit may comprise a peak detector so that, as the analyser sweeps through the selected span, the value obtained by sampling is the peak value within a frequency interval of width F. This kind of processing is not applicable with the proposed approach, so it must be assumed that any kind of detector is disabled within the instrument and straightforward sampling is carried out. In this way, (9) corresponds to the sampled trace:

$$T(kF,t) = \frac{A_0^2}{4}\left|W(kF - f_0)\right|^2 . \qquad (26)$$

Recalling that $W(f)$ is a narrow band function, estimation of sinewave power, based on (15), can be obtained by the discretised form:

$$\frac{A_0^2}{2|Z_{in}|} = K_{PSD} \cdot \int_0^{+\infty} T(f_x,t)df_x = K_{PSD} \cdot F \cdot \sum_{k=K_1}^{K_2} T(kF,t) , \quad (27)$$

where K_1F and K_2F bound the frequency interval where $|W(f - f_0)| > 0$. This further evidences the similarities with Fourier-based digital spectrum analysis. In fact, reference can be made to [7], where an accurate algorithm for parameter measurement of sinewaves and multifrequency signals, based on the use of power relationships like (27), has been proposed.

7. CONCLUSIONS

The presented approach suggests that, by taking an in-depth view of the operating principles of a scanning spectrum analyser, it is possible to unearth some general concepts, that tend to be hidden by the practical intricacies of the instrument. It is hoped that the theory introduced in this paper will also help in the understanding of newer generation instruments, where Fourier-based analysis is being progressively introduced with direct sampling of the IF filter output. The measurement procedures outlined in the paper can be applied to any kind of scanning spectrum analyser although, being different from the standard practice, they may need to be implemented as off-line processing algorithms. As long as a properly digitised trace and the required calibration data are available, results can be expected to be at least comparable.

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