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POSSIBILITY EQUIVALENCE OF UNIMODAL ASYMMETRIC PROBABILITY DISTRIBUTIONS

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Abstract – At the application level, it is important to be able to define around the measurement result an interval which will contain an important part of the distribution of the measured values, that is, a confidence interval. This practice acknowledged by the ISO Guide is a major shift from the probabilistic representation as a confidence interval represents a set of *possible* values for a parameter associated with a confidence level. It can be viewed as a probability-possibility transformations by viewing possibility distribution as encoding confidence intervals. In this paper, after having recalled probability/possibility links by the notion of confidence intervals, previous works concerning transformation of symmetric probability distribution into possibility distributions are extended to asymmetric probability distributions.

Keywords: measurement uncertainty, possibility theory, asymmetric distribution.

1. INTRODUCTION

Uncertainty is a key concept for the measurement expression [1][2][3]. Indeed, in many application domains, it is important to take the measurement uncertainties into account, especially in order to define around the measurement result an interval which will contain an important part of the distribution of the measured values, that is, a confidence interval [4]. Such an interval allows to define later decision risks, as for example the risk to accept a defective lot, the risk to exceed an alarm threshold, etc.

A main tool to deal with sensor measurement uncertainty is statistics [5]. This tool needs a mathematical support to be used, especially to propagate uncertainties. Two theories are mainly considered : the interval calculus [6] and the probability theory [5]. Although the interval calculus allows simple calculations, the resulting model is very imprecise. Moreover, it only supplies the confidence interval of the 100% confidence degree. Thus, the use of the probability theory seems to be necessary to supply confidence intervals, but to handle the whole sets of confidence intervals is quite complex by a probability approach. This practice corresponds to a major shift from the regular probabilistic representation, as a confidence interval represents a set of *possible* values for a parameter, associated with a confidence level. It can be viewed as a probability-possibility transformation, quite the converse move with respect to the Laplacean indifference principle, which presupposes uniform probability distributions when there is equipossibility among cases. However the weak point of the confidence interval approach is the necessity of choosing a confidence level. It is usually taken as 95% (which means a .05 probability for the value to be out of the interval). However this choice is rather arbitrary.

In fact, possibility measures can encode families of probability distributions [7][8] and can be viewed as a particular case of random sets [9][10]. Hence it is tempting to try to generalise the notion of confidence interval using a probability-possibility transformation. The idea of viewing possibility distributions, especially membership functions of fuzzy numbers, as encoding confidence intervals, is actually not new. Well before the advent of fuzzy sets, in the late forties, Shackle [11] has introduced the connection between confidence intervals and the measurement of possibility in his theory of potential surprise, which is a first draft of possibility theory. McCain [12] also independently pointed out that a fuzzy interval models a nested set of confidence intervals with a continuum of confidence levels. The idea of relating fuzzy sets to nested confidence sets via a probability-possibility transformation was proposed by Dubois and Prade [9]. Doing so, it is clear that some information is lost (since an imprecise probability is obtained). However it may supply a nested family of confidence intervals instead of a single one. The guiding principle for this transformation is to minimise informational loss. The corresponding transformation has already been proposed in the past [9][13][14]. More recent results have been obtained by Lasserre, Mauris et al. [15][16] and applied to the problem of representing physical measurements associated to a symmetric distribution.

This paper further explores the connection between this probability-possibility transformation, confidence intervals and the concepts developed in the ISO guide for the expression of uncertainty in measurement, as explained in section 2. In the third section, we apply the proposed probability possibility transformation to asymmetric probability distributions. Some concluding remarks points out the interest of the approach and some perspectives.

2. PROBABILITY VERSUS POSSIBILITY UNCERTAINTY REPRESENTATIONS

2.1. Basics of probability and possibility theories

Possibility measures are set functions similar to probability measures, but they rely on an axiom which only involves the operation " maximum " instead of the "addition" [17][18]. A possibility measure Π on a finite set X is characterised by a possibility distribution $\pi : X \rightarrow [0,1]$ such that :

$$\forall A \subseteq X, \Pi(A) = \sup\{\pi(x), x \in A\}. \quad (1)$$

To ensure $\Pi(X) = 1$, a normalization condition demands that $\pi(x) = 1$, for some $x \in X$

A numerical possibility distribution π: $R \rightarrow [0,1]$ is called a fuzzy interval as soon as its α-cuts $A_\alpha = \{x, \pi(x) \geq \alpha\}$ are (closed) intervals. When the modal value of π (i.e. x^* is such that $\pi(x^*) = 1$), it is also called a fuzzy number. Then, if π is continuous, $N(A_\alpha) = 1 - \alpha$, $\forall\alpha \in (0, 1]$, and

$\pi(x) = \sup \{\Pi((A_\alpha)^c), x \in A_\alpha\}$.

As it turns out, a numerical possibility measure can also be viewed as an upper probability function [19]. Formally, such a real-valued possibility measure Π is equivalent to the family $\boldsymbol{P}(\Pi)$ of probability measures such that $\boldsymbol{P}(\Pi) = \{P, \forall A \subseteq X, P(A) \leq \Pi(A)\}$.

2.2. Confidence intervals

Let p be a unimodal probability density and x^* be a "one-point" estimation of the "true" value, for example the mode or the mean value of the probability density. An interval is defined around the "one-point" estimation, and its confidence level corresponds to the probability that this interval contains the "true" value. For a confidence level α, such an interval, denoted I^*_α is called a confidence interval, and its confidence level is $P(I^*_\alpha) = \alpha$ (95%, 99% are values often used in the measurement area); $1 - P(I^*_\alpha)$ is the risk level, that is, the probability for the real value to be outside the interval. In the following, a family of such confidence intervals supposed to be closed, is assumed to be given.

2.3. Possibility representation of confidence intervals

Definition 1: A fuzzy uncertainty interval around x^* (denoted π^*) representing the continuous probability density p is the possibility distribution defined by identifying each α-cut of π^* with each closed confidence interval I^*_α of confidence level α around the nominal value x^* computed from p.

According to the definition, a possibility measure Π^* and its distribution π^* can be defined as follows:

$$\Pi^*((I^*_\alpha)^c) = 1 - P(I^*_\alpha) \quad (= 1 - \alpha),$$
$$\pi^*(x) = \sup \{1 - \alpha, x \in I^*_\alpha\}. \quad (2)$$

The possibility distribution π^* is continuous and encodes the whole set of confidence intervals in its membership function. It can be proved that indeed $p \in \boldsymbol{P}(\Pi^*)$.

Theorem 1: For any probability density p, the possibility distribution π^* in Definition 1 dominates p, that is: $\forall A$ measurable, $\Pi^*(A) \geq P(A)$, Π^* and P being the possibility and probability measures associated respectively to π^* and p.

Proof : For any measurable set A $\subseteq \boldsymbol{R}$, define the set $C = \{x \in \boldsymbol{R}, \pi^*(x) \leq \Pi^*(A)\}$. Obviously, $A \subseteq C$, because $\forall A$ measurable, $\Pi^*(A) = \sup_{x \in A} \pi^*(x) = \Pi^*(C)$. Now, $P(C) = \Pi^*(A)$. Indeed C^c is the cut of level $\Pi^*(A)$ of π^*, therefore, $P(C^c) = 1 - \Pi^*(A)$, due to definition 1. Finally, $\Pi^*(A) \geq P(A)$ since $A \subseteq C$.

In the sequel, we show that ensuring the preservation of the maximal amount of information in π^* can motivate the choice of the nominal value as the mode x^m of the probability density. This is justified by the following lemma. In this lemma, the length of a measurable subset of the reals is its Lebesgue measure.

Lemma 1: For any continuous probability density p having a finite number of modes, any minimal length measurable subset I of the real line such that $P(I) = \alpha \in (0, 1]$, is of the form $\{x, p(x) \geq \beta\}$ for some $\beta \in [0, pmax]$ where $pmax = \sup_x p(x)$. It thus contains the modal value(s) of p.

Proof : Let $I = \{x, p(x) \geq \beta\}$. I is a closed interval or a finite union thereof. Assume that there exists another measurable subset J of R such that $P(J) = P(I)$ with $length(J) < length(I)$. Considering the three following disjoint domains of R: $I \cap J$, $I \backslash J$ and $J \backslash I$, we find that since $P(J) = P(I)$ by assumption: $P(J) - P(I) = \int_{I \backslash J} p(x)dx - \int_{J \backslash I} p(x)dx$. Now, for x $\in I \backslash J$, $p(x) \geq \beta$, and for x $\in J \backslash I$, $p(x) < \beta$,, therefore: $\int_{I \backslash J} dx = length(I \backslash J) \leq \int_{J \backslash I} dx = length(J \backslash I)$. Hence, $length(I \backslash J) + length(I \cap J) = length(I) \leq length(J \backslash I) + length(I \cap J) = length(J)$ which contradicts the assumption.

Note: the lemma 1 can be easily extended to continuous probability distributions of $\boldsymbol{R}^d$ by replacing in the above proof the length by the Lebesgue measure on $\boldsymbol{R}^d$, i.e the hyper-volume.

This lemma has been proved in [20] for unimodal probability densities. Here, the proof is valid for any continuous probability density with a finite number of modes. However the unicity of the minimal length set I_α such that $P(I_\alpha) = \alpha \in (0, 1]$ is not always ensured. It exists for unimodal continuous probability densities with no range of constant value. It is also obvious from lemma 1 that for any confidence level α, the smallest sets I_α such that $P(I_\alpha) = \alpha \in (0, 1]$ are nested. The lemma proves that these most informative confidence sets are cuts of the probability density. The corresponding possibility distribution is denoted π^{xm} and $\pi^{xm}(x) = \sup \{1 - \alpha, x \in I_\alpha\}$.

Since the minimal length sets I_α contain the modal values of p, *i.*e. x^m such that $p(x^m) = pmax$, whatever the probability density, it gives a justification for choosing $x^* = x^m$ and building the confidence intervals around modal values even for asymmetrical or multi-modal densities. Choosing confidence sets of minimal length ensures that this possibility distribution will be maximally specific. The degree of imprecision of π is defined by $\int_R \pi(y)dy = \int_{[a,\ b]} \pi(y)dy$ (if $[a, b]$ is the support of the unimodal density p). It is also equal, due to Fubini's theorem, to $\int_{[0,\ 1]} length(A_\alpha)d\alpha$. Thus, minimising the size of the cuts of π dominating p comes down to minimising the imprecision of π.

This theorem and this lemma has been applied in [20] to symmetric distributions. Hereafter, the asymmetric case is considered.

3. POSSIBILITY EQUIVALENT OF ASYMMETRIC PROBABILITY DISTRIBUTIONS

A closed form expression of the possibility distribution induced by confidence *intervals* around the mode $x^* = x^m$ is obtained for unimodal continuous probability densities strictly increasing on the left and decreasing on the right of x^m by :

$$\forall x \in [-\infty, x^m],\ \pi^{xm}(x) = \pi^{xm}(f(x)) = \int_{(-\infty,\ x]} p(y)dy + \int_{(f(x),\ +\infty]} p(y)dy \qquad (3)$$

where f is the mapping defined by: $\forall x \in [-\infty, x^m],\ f(x) = y \geq x^m$ such that $p(x) = p(y)$. The function f is continuous and strictly decreasing, therefore a one-to-one mapping, and from (3) is clear that is π^{xm} is continuous too, and even differentiable, since p is continuous.

If p is symmetric with mode x^m, then the possibility distribution $\pi^{xm}(x)$ is then easily defined as :

$$\forall x \in [-\infty, x^m],\ \pi^{xm}(x) = \pi^{xm}(2x^m - x) = 1 - P([x, 2x^m - x])$$

For asymmetric distributions, (3) cannot be reduced and has to be computed in two parts. Let us consider a situation proposed in the ISO guide concerning the measurement of a vertical fixed height h of a liquid column in a manometer. The measurement system axis can shift from the vertical of a small angle β. The determined distance l will be always superior to h, because $l=h\cos\beta$. If we introduce $d=1-\cos\beta$, we obtain $h=l(1-d)$. If the angle is small we have: $p(d)=(1/\sigma\sqrt{\pi d})\exp(-d/\sigma^2)$ with $d>0$.

The equivalent possibility distribution obtained by the formula (3) is plotted in the following figure for $\sigma=1$.

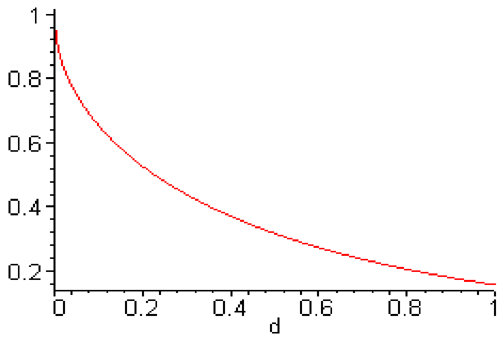

Fig1. Example of an asymmetric distribution

In [20], it is demonstrated that the triangular possibility distribution is an optimal transform of the uniform probability distribution and it is the upper envelope of all the possibility distributions transformed from symmetric probability densities with the same support. Unfortunately this result does not carry over to asymmetric distributions.

Counter-example: let us consider for example the piecewise linear probability density of support $[-2, +2]$ defined by :

$\forall x \in [-2, -1.5],\ p(x) = 0.6x + 1.2,$

$\forall x \in [-1.5, 0]\ p(x) = (0.2/3)x + 0.4\ ,$

$\forall x \in [0, 2],\ p(x) = -0.2x + 0.4.$

A piece-wise parabolic possibility distribution is obtained by applying equation (3) (see figure 1). It gives $\pi^{xm}(-1.5) = 0.3 > \pi_{triangle}(-1.5)=0.25.$

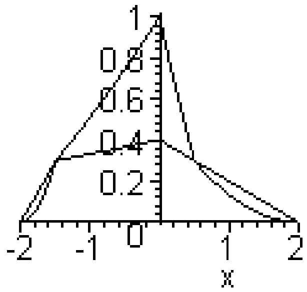

Fig2. Example of asymmetric distribution not dominated by the asymmetric triangular one

4. CONCLUSION

This paper has proposed a systematic approach for the transformation of a continuous probability distribution into a maximally specific possibility distribution that enables upper bounds of probabilities of events to be computed. The obtained possibility distribution encodes a nested family of tightest confidence intervals around the mode of the statistical distribution considered. Applied to asymmetric densities, this result provides an original way of representing them by a possibility approach.

This approach will help for the uncertainty propagation represented by symmetric distributions because operations like division do not preserve the symmetry of distributions.

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