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IMPROVING SINE-FITTING ALGORITHMS FOR AMPLITUDE AND PHASE MEASUREMENTS

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Abstract − In this paper, improved sine-fitting algorithms for the measurement of amplitude and phase difference between two records of digitized sine waves with the same frequency are presented. These algorithms can be used for example in impedance measurements or to accurately measure the input and the output of a linear system to be characterized in the frequency domain both in amplitude and in phase.

Keywords: sine-fitting algorithms, amplitude and phase measurement.

1. INTRODUCTION

To determine the frequency response of a given system, a usual method consists on sweeping the input frequency while measuring the input/output amplitude ratio as well as the phase difference. To achieve a reasonable accuracy using a data acquisition board, some processing as to be done on the acquired records. In [1,2], two basic numerical methods are described to perform sine-fitting in order to characterize a sine wave acquired by a digitizer. These two numerical methods are known as the three and the four-parameter sine-fitting algorithms.

The first algorithm determines the amplitude, phase and DC component of the acquired signal in one, non-iterative step for a given frequency. Its major drawback is that, when the relation between the input and sampling frequencies is not correctly known, the results of the method are not reliable.

In the four-parameter sine-fitting algorithm, a first estimation of the frequency is used to determine the initial estimations of the amplitude, phase and DC component. Afterwards, an iterative method is used to correct the frequency. Although, this method is an improvement of the three-parameter algorithm, it is nonlinear and does not guarantee convergence. In fact, there are some cases where the algorithm converges to local minimums resulting in incorrect parameters [3]. One of the options to get better results from the four-parameter algorithm is to improve the initial frequency estimation. To achieve this, the three-parameter algorithm with increasing number of points and a linear-regression technique can be used [3,4], the IpDFT can be applied to the sampled records [5,6] or a frequency estimator based on a total least-square method can be used [7].

However, these sine-fitting techniques do not take into account the specific restriction of the present problem (*i.e.*, the records are the result of the digitalization of two signals with the same frequency).

In this paper different algorithms (based on the three and four-parameter and a new seven-parameter) that can reduce the uncertainty of the final results are analyzed.

2. SINE-FITTING

The two sine waves are digitized at a sampling frequency, f_S, each having M equally spaced points and a common frequency f.

The M digitized values are: $y_{1,1} \ldots y_{1,M}$ for the first channel and $y_{2,1} \ldots y_{2,M}$ for the second channel. Each sample is acquired at $t_{k,m}$ where $k=1,2$ is the channel number and $m=1\ldots M$ is the sample number. Note that $t_{k,m+1}-t_{k,m}=T_S=1/f_S$ although $t_{1,m}$ can, and usually is, different from $t_{2,m}$ due to the way the digitizer samples multiple channels.

The sine-fitting methods described in [1,2], minimize the residual least-square error

$$\varepsilon_k=\sum_{m=1}^{M}\left[y_{k,m}-A_k\cos\left(2\pi f_k t_{k,m}\right)-B_k\sin\left(2\pi f_k t_{k,m}\right)-C_k\right]^2 \quad (1)$$

for channel k, where A_k and B_k define the amplitude and phase of the sine wave, C_k is the DC component and f_k is the detected frequency. The minimum value of the error depends on the noise present in the acquired signals but also on its distortion – *i.e.*, signal harmonics and spurious components which can be present in the input signal or introduced by the acquisition channel.

The matrix used in iteration i to apply the least-square procedure in the four-parameter algorithm is [1, p. 22]

$$\mathbf{D}_{k,i}=\begin{bmatrix} w\left(f_{k,i},t_{k,1}\right) & g\left(f_{k,i},t_{k,1}\right) & 1 & h\left(A_{k,i-1},B_{k,i-1},f_{k,i},t_{k,1}\right)\\ w\left(f_{k,i},t_{k,2}\right) & g\left(f_{k,i},t_{k,2}\right) & 1 & h\left(A_{k,i-1},B_{k,i-1},f_{k,i},t_{k,2}\right)\\ \vdots & \vdots & \vdots & \vdots\\ w\left(f_{k,i},t_{k,M}\right) & g\left(f_{k,i},t_{k,M}\right) & 1 & h\left(A_{k,i-1},B_{k,i-1},f_{k,i},t_{k,M}\right)\end{bmatrix} \quad (2)$$

where

$$\begin{aligned} w(f,t) &= \cos(2\pi ft) \\ g(f,t) &= \sin(2\pi ft) \\ h(A,B,f,t) &= -At\sin(2\pi ft) + Bt\cos(2\pi ft). \end{aligned} \tag{3}$$

If this algorithm is applied to the two records independently, it will result in the estimation of two different frequencies for the sine waves and increase the overall uncertainty.

In the new seven-parameter sine-fitting technique here presented, both records are used to determine all the parameters of the sine waves. The total minimized error is

$$\varepsilon = \sum_{m=1}^{M}\left[y_{1,m} - A_1\cos(2\pi ft_{1,m}) - B_1\sin(2\pi ft_{1,m}) - C_1\right]^2 + \sum_{m=1}^{M}\left[y_{2,m} - A_2\cos(2\pi ft_{2,m}) - B_2\sin(2\pi ft_{2,m}) - C_2\right]^2. \tag{4}$$

The matrix used in each iteration has $2M$ lines and 7 columns but at least $6M$ elements are zero:

$$\mathbf{D}_i = \left[\begin{array}{c|c|c} \mathbf{Q}_{1,i} & \mathbf{R}_{1,i} & 0 \\ \hline 0 & \mathbf{R}_{2,i} & \mathbf{Q}_{2,i} \end{array}\right] \tag{5}$$

with

$$\mathbf{Q}_{k,i} = \begin{bmatrix} w(f_i,t_{k,1}) & g(f_i,t_{k,1}) & 1 \\ w(f_i,t_{k,2}) & g(f_i,t_{k,2}) & 1 \\ \vdots & \vdots & \vdots \\ w(f_i,t_{k,M}) & g(f_i,t_{k,M}) & 1 \end{bmatrix} \tag{6}$$

and

$$\mathbf{R}_{k,i} = \begin{bmatrix} h(A_{k,i-1},B_{k,i-1},f_i,t_{k,1}) \\ h(A_{k,i-1},B_{k,i-1}f_i,t_{k,2}) \\ \vdots \\ h(A_{k,i-1},B_{k,i-1}f_i,t_{k,M}) \end{bmatrix}. \tag{7}$$

The final column vector

$$x = \begin{bmatrix} A_1 & B_1 & C_1 & f & A_2 & B_2 & C_2 \end{bmatrix}^T, \tag{8}$$

is obtained by

$$x = \left[\mathbf{D}^T\mathbf{D}\right]^{-1}\left[\mathbf{D}^T y\right]. \tag{9}$$

where $\mathbf{D}$ is the matrix (5) of the last iteration of the method and $\mathbf{D}^T$ is its transpose.

3. NUMERICAL RESULTS

The simulated ADC has an input range of ±1V, 12 bit resolution and a time delay of $T_S/2$ between the sampling of both channels – *i.e.*, $t_{2,m} - t_{1,m} = T_S/2$, to match the ADC used to acquire the experimental data. Gaussian noise is added to the sine waves prior to digitizing to simulate the real system conditions where noise is always present. For each situation, 10000 different tests were performed and the corresponding average and standard deviation values of the sine wave parameters were determined. The sampling frequency is $f_S \simeq 24.39\,\text{kS/s}$ which corresponds to $T_S = 41\,\mu\text{s}$ and the input frequency is $f_{IN} = 1\,\text{kHz}$.

In all the sine-fits performed in this paper, the criteria to stop the iterative method is that the relative frequency correction is below 10^{-7} – which corresponds to $\Delta f_i < 10^{-10}\,\text{Hz}$ for $f_{IN} = 1\,\text{kHz}$. The initial frequency estimation is obtained with the IpDFT [5,6].

3.1. Single-channel sine-fitting

Starting with the standard four-parameter sine-fitting algorithm it is important to understand the influence of the amplitude of the sine wave and noise in the determination of the signal frequency.

In Fig. 1 the relative standard deviation of the detected frequency, σ_f, is shown as a function of the input sine wave amplitude and noise RMS value n_{RMS}. The standard deviation decreases with the increasing of the amplitude even for amplitudes two times the value of the ADC range (in these situations, the samples corresponding to the ADC saturation are not used, *i.e.*, although M samples are acquired for each channel, the number of points used in the sine-fitting procedure is smaller). As the noise increases so does the standard deviation of the estimated frequency.

Using only 122 points, the frequency can be determined with an uncertainty of 0.1 Hz (0.01%) for the higher amplitudes even with $n_{RMS} = 10\,\text{mV} \simeq 21\,\text{LSB}$.

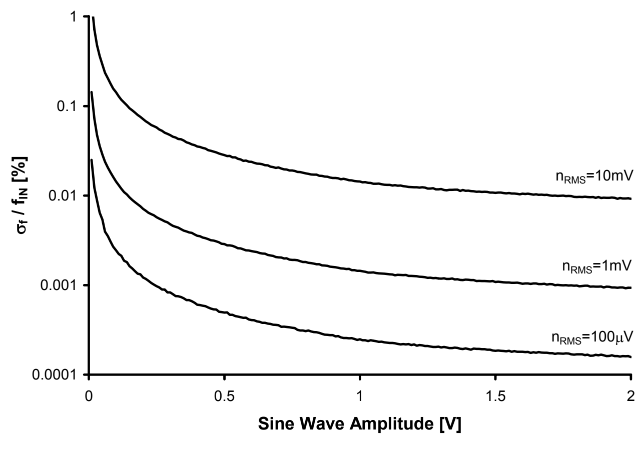

Fig. 1 Relative standard deviation of the frequency computed by the four-parameter algorithm with f_{IN} = 1kHz, f_S = 24.39kS/s and 122 samples. For these parameters, approximately 5 periods of the sine wave are acquired.

3.2. Four-parameter sine-fitting for each channel

The first option to perform sine-fitting in two records with the same frequency, in order to determine their amplitudes and phase difference is to apply the four-parameter algorithm to each record. In Fig. 2 the standard deviation of the phase difference is shown as a function of the amplitude of one of the records for four particular values of the amplitude of the other record. The standard deviation of the phase difference decreases with the increasing amplitude of the channels. Nevertheless, these results hide the fact that the estimated frequency of the records is not the same although this difference is reflected in higher standard deviations of the determined phase difference.

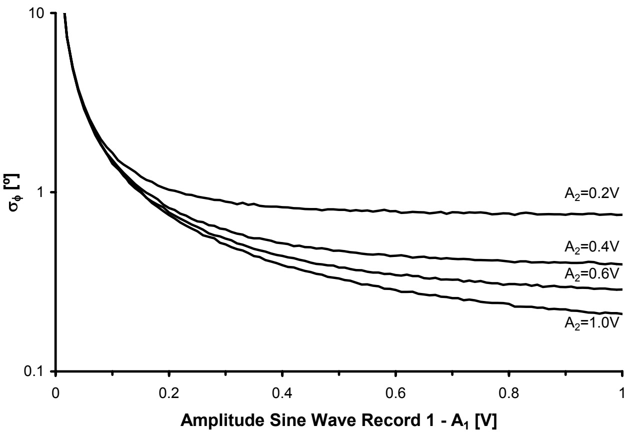

Fig. 2 Standard deviation of the phase difference obtained by using two four-parameter algorithms and 10mV RMS noise. All other parameters as in Fig. 1.

3.3. Combined four- and three-parameter sine-fitting

Since the detected frequency depends on the amplitude of the sine wave (due to the presence of noise and ADC quantization – Fig. 1), the first approach to improve the results consists in applying the four-parameter algorithm to the record with higher amplitude and use the resulting frequency as input for a three-parameter algorithm applied in the remaining record – Fig. 3. To determine the channel with the highest amplitude, a three-parameter sine-fit is first applied to each record.

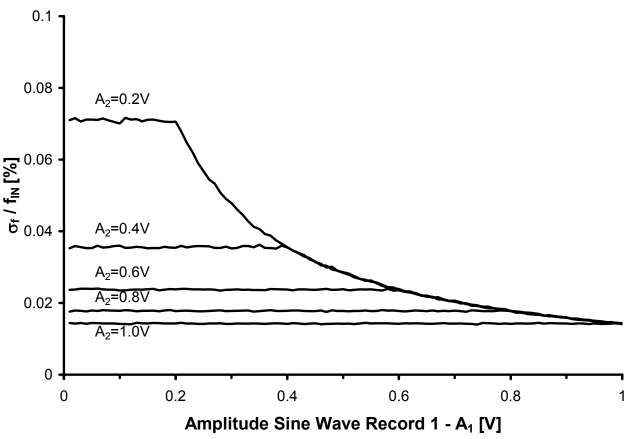

Fig. 3 Relative standard deviation of the frequency obtained with the combined four- and three-parameter algorithm. The results correspond to the situation of Fig. 2.

The standard deviation of the frequency depends on the amplitude of the highest amplitude channel, since it is this record that is used in the four-parameter algorithm. For example, when $A_1 < 0.2\,\text{V}$ and $A_2 = 0.2\,\text{V}$, the record of the first channel is not used to determine the common frequency and therefore the standard frequency deviation remains constant. For $A_1 > 0.2\,\text{V}$, the channel with the highest amplitude is used to compute the frequency and the standard deviation of the frequency is reduced when its amplitude increases.

3.4. Seven-parameter sine-fitting

In Fig. 4, the results obtained by applying the seven-parameter method demonstrate that the standard deviation of the frequency is no longer limited for lower amplitudes as in Fig. 3. In fact, for $A_1 < 0.2\,\text{V}$ and $A_2 = 0.2\,\text{V}$, the standard deviation of the frequency is lower because both records are used to determine the common frequency and all the other parameters of the sine waves.

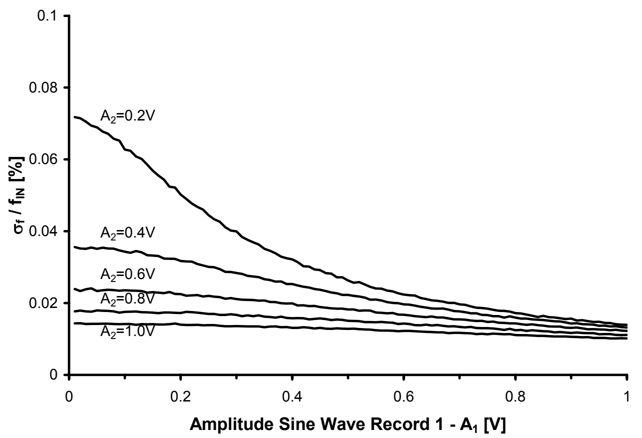

Fig. 4 Relative standard deviation of the frequency obtained with the seven-parameter algorithm. All parameters as in Fig. 2.

In Fig. 5, the standard deviation of the phase difference is plotted. The results of the seven-parameter algorithm show a decrease of the standard deviation of approximately 50% in comparison with the results obtained by using two four-parameter algorithms – Fig. 2.

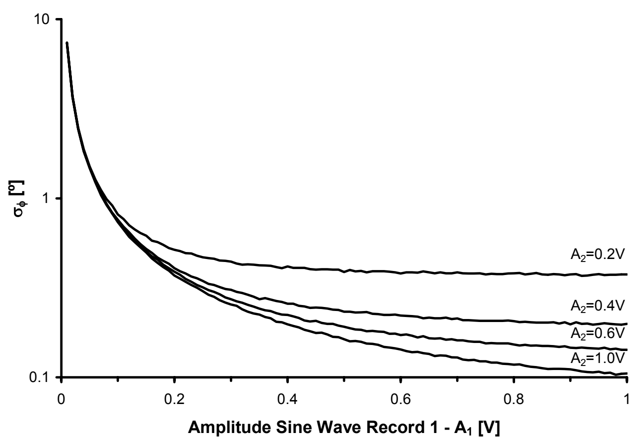

Fig. 5 Standard deviation of the phase difference obtained with the seven-parameter algorithm. All parameters as in Fig. 2.

4. EXPERIMENTAL RESULTS

The experimental results were obtained using a Keithley DAS–1601 board with 12 bit resolution in the range $\pm 1\,\mathrm{V}$. The input signal, generated by a HP33120A at $1\,\mathrm{kHz}$, is applied to two resistances in series in order to produce the different amplitudes of the channels. For each experimental point in Fig.s 6-12, 1000 acquired records are used to compute the standard deviations.

4.1. Four-parameter sine-fitting for each channel

In Fig. 6, the results obtained by applying the four-parameter algorithms to both channels independently are shown. Comparing these results with Fig. 2, it can be seen that they are similar but with a discrepancy in the amplitude of the standard deviations. This difference is due to the high noise ($10\,\mathrm{mV}$) considered in the simulation results. The estimated noise present in both channels is $130\,\mu\mathrm{V}$.

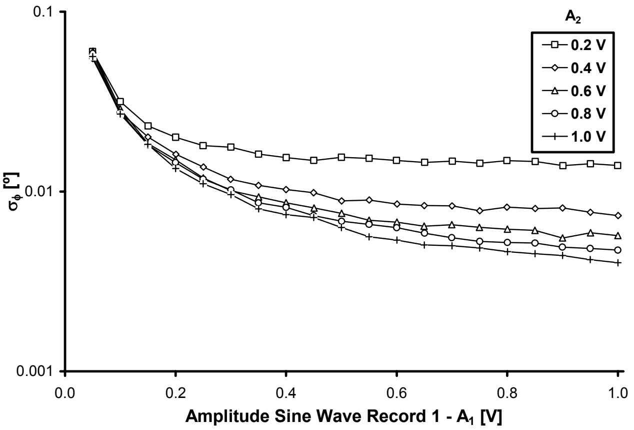

Fig. 6 Standard deviation of the phase difference obtained with the four-parameter algorithms for each channel. 122 samples acquired at f_S = 24.39kS/s with f_{IN} = 1kHz.

In Fig. 7, the comparison of the simulated results and the experimental results is shown for $A_2 = 0.2\,\mathrm{V}$.

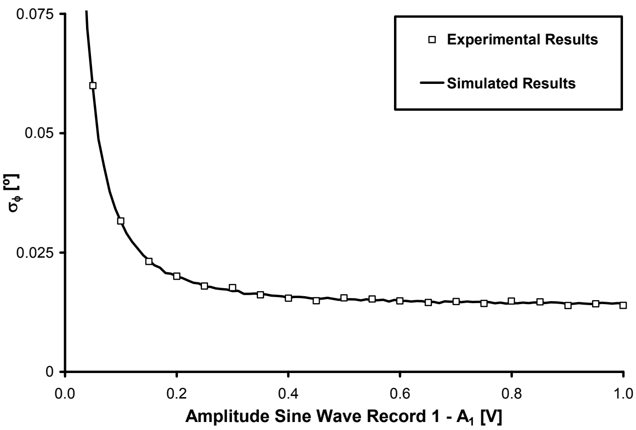

Fig. 7 Comparison of the experimental and simulation results of the standard deviation of the phase difference obtained with the four-parameter algorithms for each channel. All parameters as in Fig. 6. The simulated noise corresponds to 130 μV and the amplitude of the second channel is A_2 = 0.2 V.

4.2. Combined four- and three-parameter sine-fitting and seven-parameter fit

Fig. 8 shows the experimental results obtained with the combined four- and three-parameter fit and with the seven-parameter fit for a fixed amplitude in one of the channels. It can be seen that, the seven-parameter algorithm can reduce the uncertainty in the frequency when the amplitudes of both channels are similar. These results confirm the simulation results – Fig. 3 and Fig. 4.

The seven-parameter uses all the acquired records to better estimate the common frequency, while the combined four- and three-parameter fit uses only the record corresponding to the highest amplitude channel. When the amplitude of one channel is much higher than the amplitude of the other, both methods produce identical results. This is due to the fact, that the samples corresponding to the highest amplitude channel carry more useful information for the determination of the common frequency; therefore the combined method can achieve similar results.

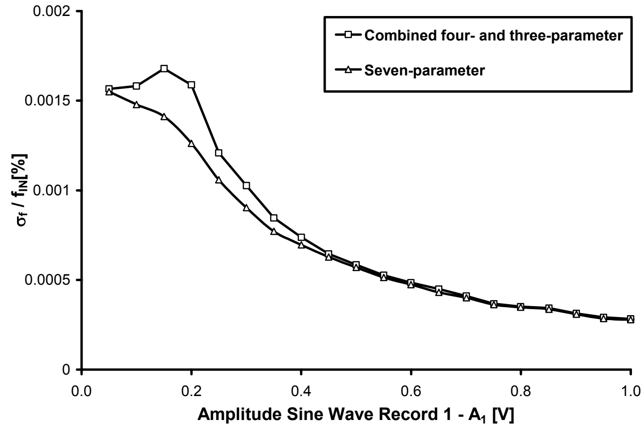

Fig. 8 Relative standard deviations of the frequency obtained with the four- and three-parameter algorithm and with the seven parameter algorithm. All parameters as in Fig. 6 with A_2 = 0.2 V.

In Fig. 9, the experimental results of the standard deviation of the phase difference between both channels obtained with the seven-parameter algorithm are shown for constant amplitude in one channel.

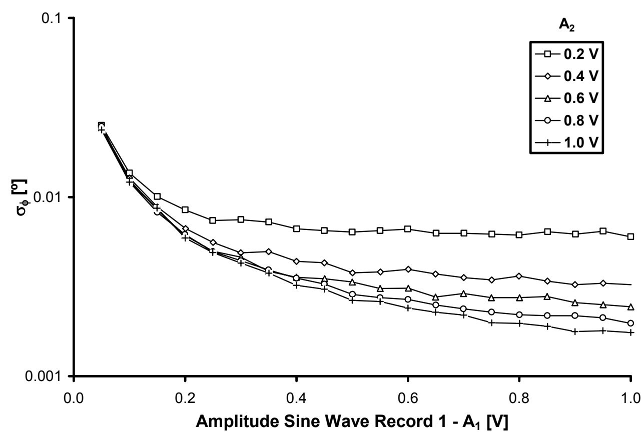

Fig. 9 Experimental results of the standard deviation of the detected phase obtained with the seven-parameter algorithm with 122 samples acquired at f_S = 24.39kS/s with f_{IN} = 1kHz.

The results again demonstrate the influence of the amplitude of the sine waves on the uncertainty of the final results. For higher amplitude signals the sine-fitting results improve to the point where the relative phase has uncertainty below 0.01º even with only 122 points acquired for each channel.

In Fig. 10, the experimental results obtained for the standard deviation of the frequency with increasing number of points per channel are shown. For records with 1000 points (41 periods), standard deviations of the phase below 0.0025º can be obtained.

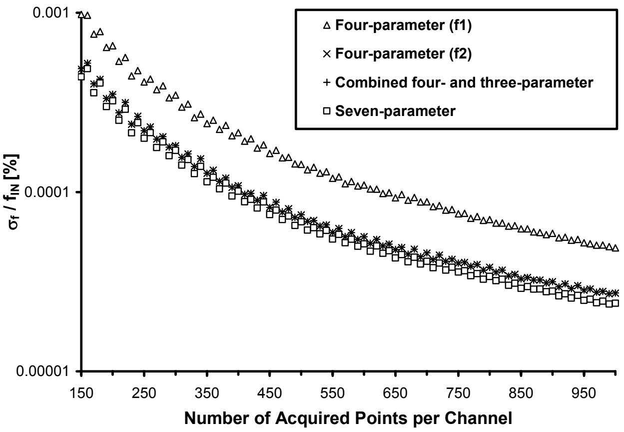

Fig. 10 Experimental results of the relative standard deviation of the frequency for the seven-parameter fit and the combined four- and three-parameter fit as a function of the number of samples with f_S = 24.39kS/s, f_{IN} = 1kHz, A_2 = 0.2V and A_1 = 0.4V.

In Fig. 11, the measured phase obtained with the seven-parameter and the four-parameter fits is shown. The error bars correspond to a coverage factor of $2\sigma_\phi$. Although the average value of both methods is similar, the standard deviation obtained with the seven-parameter is about half the standard deviation obtained when the four-parameter sine-fitting is applied to both records.

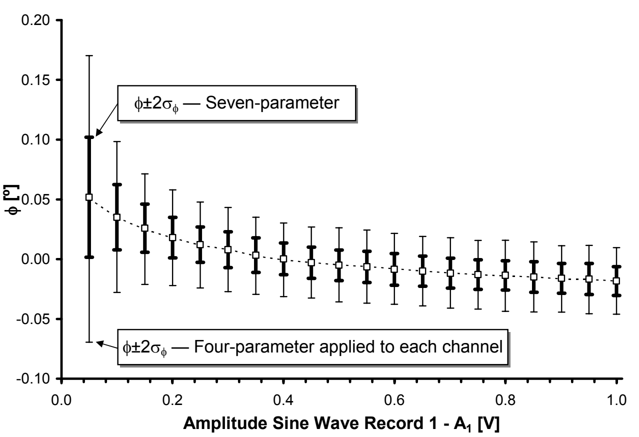

Fig. 11 Experimental results of the phase difference with the seven-parameter and the four-parameter fits as a function of the amplitude of the sine wave in the first channel with A_2 = 0.2V and 122 samples. The vertical error bars have a amplitude of $4\sigma_\phi$ for each case.

4.3 Computation time

Since the calculations were performed in LabVIEW and the algorithms are iterative, it is not possible to determine the computational time of the methods for comparison. However, after performing 10000 simulations, the relative computational time of the three proposed approaches was determined and is shown in Fig. 12 as a function of the number of acquired points.

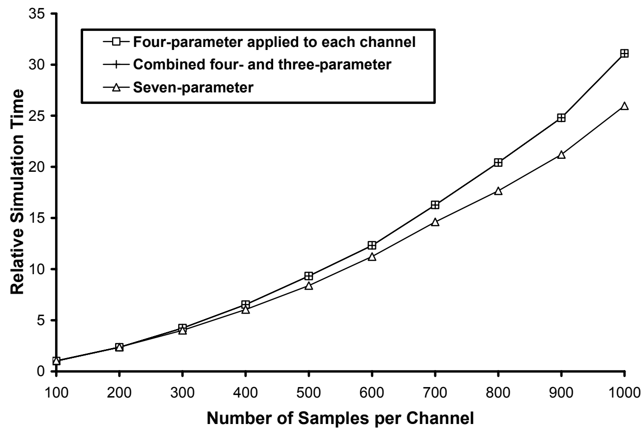

Fig. 12 Relative calculation time of the three methods examined as a function of the number of samples with f_S = 24.39kS/s, f_{IN} = 1kHz, A_2 = 0.2V and A_1 = 0.4V. All the simulation times were divided by the shortest simulation time which corresponds to the combined three- and four-parameter technique with 100 points per channel.

To compute (9), six steps are needed: (i) create matrix $\mathbf{D}$; (ii) calculate $\mathbf{D}^T$; (iii) determine $\mathbf{D}^T\mathbf{D}$; (iv) invert $\mathbf{D}^T\mathbf{D}$; (v) calculate $\mathbf{D}^T y$; (vi) multiply $\left[\mathbf{D}^T\mathbf{D}\right]^{-1}$ by $\mathbf{D}^T y$. The time to execute this task depends on the dimension of matrix $\mathbf{D}$ and is represented by $T_{LS}(l,c)$ where l is the number of lines and c the number of columns. The matrix inversion step is applied to matrixes with $c\times c$ elements – *i.e.*, the number of elements is small and does not depend on the number of points per channel.

Assuming that there are Y iterations, in the method where the four-parameter algorithm is applied to both records, two IpDFT's are computed and $2Y$ iterations with matrixes of $M\times4$ are performed. The total time is therefore

$$T_{2\times4} = 2T_{\mathrm{IpDFT}}(M) + 2Y\,T_{LS}(M,4)\,, \qquad (10)$$

where $T_{\mathrm{IpDFT}}(M)$ is the time needed to compute the IpDFT with M points. In the combined four- and three-parameter method, the total time is

$$T_{1\times4+1\times3} = T_{\mathrm{IpDFT}}(M) + 3\,T_{LS}(M,3) + Y\,T_{LS}(M,4)\,. \qquad (11)$$

Three three-parameter sine-fits are needed (two to estimate the channel with the highest amplitude and one to determine the parameters of the channel with the lowest amplitude).

In the seven-parameter method, the time is

$$T_{1\times 7} = T_{\text{IpDFT}}(M) + Y\, T_{LS}(2M, 7) \,. \qquad (12)$$

The fact that the seven-parameter method takes less time to compute may seem strange but it is justified by the fact the initial frequency estimation (obtained with the IpDFT) is very good, resulting in a small the number of iterations (usually less than three). In these conditions (caused not only by the IpDFT estimation but also by the low noise), the execution time of the seven-parameter method is exceeded by the overall iterations of the other methods.

5. CONCLUSIONS

Numerical simulation and experimental results demonstrate the need for sine-fitting algorithms that take into account physical restrictions of the acquired records – in this case the common frequency. The proposed algorithm consists of a seven-parameter sine-fitting algorithm that can reduce by half the phase difference uncertainty of the four-parameter algorithm.

This method can be applied to the measurement of impedances as described in [8] to further reduce the uncertainty of the final results.

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