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IN–BAND POWER ESTIMATION OF WINDOWED DELTA–SIGMA SHAPED NOISE

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Abstract – Performances of $\Delta\Sigma$ modulators are evaluated by applying a coherently sampled tone and by estimating powers of the in–band tones and noise. In particular, the power of the shaped noise is usually estimated by subtracting the evaluated input tone from the output data and by integrating the power spectral density estimated by means of the periodogram. Although the coherency, the finite number of processed samples induces spectral leaking of the wide–band noise, thus affecting the noise power estimate. To cope with such an issue, usually data are weighted by the Hanning sequence. In this paper, the noise power estimation error induced by the use of such window is investigated, and a criterion for choosing the minimum number of samples N which bounds the relative leakage error within a specified maximum value is explicitly given. Moreover, it is shown that, for any N, such an error is negligible for modulator orders lower than 3. Higher order modulators require the use of a large number of samples to bound the relative error of the noise power estimate when high oversampling ratios are employed.

Keywords: Delta–Sigma spectrum, spectral leakage, Hanning window.

I. INTRODUCTION

Figures of merit used for evaluating performances of $\Delta\Sigma$ modulators are based on the processing of the output data by means of algorithms which estimate the powers of the tones and of the shaped quantization noise in the band of interest. In order to properly measure such quantities, usually algorithms defined in the amplitude– or in the time–domain are employed under coherent sampling conditions, as recommended by the IEEE standards 1241 and 1057 and by the European draft standard Dynad [1]-[3]. Whatever the adopted estimation technique, because of the finite length of the acquired samples, the spectral leakage of the shaped quantization noise, which appears in the low frequency region also when coherency is applied, affects the estimation of the corresponding power [4]. To reduce the effect of this phenomenon, the output bit–stream is usually weighted by a suitable sequence. The commonly employed window is the Hanning sequence, which has been demonstrated to be the optimal two–term cosine window for reducing the spectral leakage of both narrow– and wide–band components [4]. However, on the basis of the modulator order and of the band of interest, the employed window affects the accuracy of the estimates of the shaped quantization noise power, especially when high oversampling ratios (OSRs) are employed.

In practical cases, thermal noise and non–idealities of the components introduced by the fabrication process, represent noise sources with a non–negligible in–band power with respect to the quantization noise. In particular, the most critical block is the first integrator loop, since noise sources introduced by this stage experience the input signal transfer function. As a consequence, a certain number of trade-offs have to be taken into account in order to achieve optimal performance.

It is then useful analyze the theoretical limits on the estimation of the noise power induced by the spectral leakage of the wide–band noise. In this paper, the error induced on the Hanning windowed noise power estimate is investigated. In particular, a criterion for choosing the minimum number of samples which guarantees the relative error estimation bounded within a given maximum value is given. Moreover, it is demonstrated that the Hanning sequence is the optimal window only for modulator orders lower than three.

II. WIDE–BAND NOISE SPECTRAL LEAKING

The in–band output noise power of $\Delta\Sigma$ modulators is often measured by integrating the averaged power spectral density (PSD), estimated by means of the periodogram, in the normalized frequency interval $[0, 1/2OSR[$. Since the periodogram is calculated using a finite number of acquired samples N, output data are usually weighted by a suitable window. By modeling the internal quantizer as an additive white noise with zero–mean and variance equal to σ_e^2, the analyzed output signal of an ideal modulator can then be written as:

$$y_{Lw}[n] \triangleq y_L[n]w[n] = (x[n-L] + q_L[n])w[n], \qquad n = 0, ..., N-1. \tag{1}$$

where $q_L[\cdot]$ is the quantization error of the L–th order shaped modulator, $x[\cdot]$ is the input signal and $w[\cdot]$ is the employed window sequence normalized to the square root of its energy value in order to bound the maximum value of the window autocorrelation to 1.

Fig. 1(a) shows the periodogram, $\widehat{P}_{qLw}(f_k)$, of a 1–bit $\Delta\Sigma$ converter assuming various modulator orders, as indicated by the corresponding labels. Output data have been windowed by the Hanning sequence, and the periodogram has been evaluated on 50 non–overlapped records, each of length $N = 2^{10}$. Moreover, the discrete frequency axis, f_k, has been normalized to the modulator sampling frequency, i.e. $f_k = k/N, k = 0, ..., N/2-1$. Spectral leaking of the wide–band component clearly manifests itself in the low–frequency region for the fourth–order modulator, while lower order modulators seem not to be affected by such a phenomenon.

In order to quantify the error of the in–band power estimate, the PSD of the windowed shaped noise $q_{Lw}[\cdot] \triangleq q_L[n]w[n]$, $P_{q_Lw}(f)$, has been calculated by applying a Discrete Time Fourier Transform to its autocorrelation sequence [5], as indicated in App. A, thus obtaining :

$$P_{q_Lw}(f) = P_{q_L}(f) + E_{wq_L}(f), \qquad |f| < 0.5 \tag{2}$$

where $P_{qL}(f) = \sigma_e^2 2^{2L}\sin^{2L}(\pi f)$ is the PSD of the L–order shaped quantization noise, f is the normalized frequency and $E_{wq_L}(f)$ is the contribution of the employed window. Such a contribution has been calculated for $L = 1, ..., 4$ for the Hanning window as reported in App. A, and the resulting expressions are:

$$E_{wq_1}(f) \simeq 2\sigma_e^2\frac{2}{3}\frac{\pi^2}{N^2}\left(1-\frac{1}{3}\frac{\pi^2}{N^2}\right)\cos(2\pi f) \tag{3}$$

$$\begin{aligned} E_{wq_2}(f) \simeq\ & 2\sigma_e^2\frac{8}{3}\frac{\pi^2}{N^2}\left(\left(1-\frac{1}{3}\frac{\pi^2}{N^2}\right)\cos(2\pi f)\ + \right.\\ & \left. -\left(1-\frac{4}{3}\frac{\pi^2}{N^2}+\frac{\pi^2}{N^3}\right)\cos(4\pi f)\right) \end{aligned} \tag{4}$$

$$\begin{aligned} E_{wq_3}(f) \simeq\ & 2\sigma_e^2\frac{\pi^2}{N^2}\left(5\left(1-\frac{1}{3}\frac{\pi^2}{N^2}\right)\cos(2\pi f)\ + \right.\\ & +16\left(1-\frac{4}{3}\frac{\pi^2}{N^2}+\frac{\pi^2}{N^3}\right)\cos(4\pi f) + \\ & \left. +\left(6-18\frac{\pi^2}{N^2}+\frac{64}{3}\frac{\pi^2}{N^3}\right)\cos(6\pi f)\right) \end{aligned} \tag{5}$$

$$\begin{aligned} E_{wq_4}(f) \simeq\ & 2\sigma_e^2\frac{\pi^2}{N^2}\left(56\frac{2}{3}\left(1-\frac{1}{3}\frac{\pi^2}{N^2}\right)\cos(2\pi f)\ + \right.\\ & -28\frac{8}{3}\left(1-\frac{4}{3}\frac{\pi^2}{N^2}+\frac{\pi^2}{N^3}\right)\cos(4\pi f) + \\ & +8\left(6-18\frac{\pi^2}{N^2}+\frac{64}{3}\frac{\pi^2}{N^3}\right)\cos(6\pi f) + \\ & \left. -\frac{8}{3}\left(4-\frac{64}{3}\frac{\pi^2}{N^2}+\frac{34}{3}\frac{\pi^2}{N^3}\right)\cos(8\pi f)\right) \end{aligned} \tag{6}$$

Dash-bolded and bolded lines in Fig. 1(b) represent the PSD functions $P_{qL}(f)$ and $P_{q_Lw}(f)$, respectively, of a 1–bit $\Delta\Sigma$ modulator with a full–scale (FS) equal to 1 by assuming the Hanning window of length $N = 2^{10}$, and $L = 1, ..., 4$, as indicated by the corresponding labels.

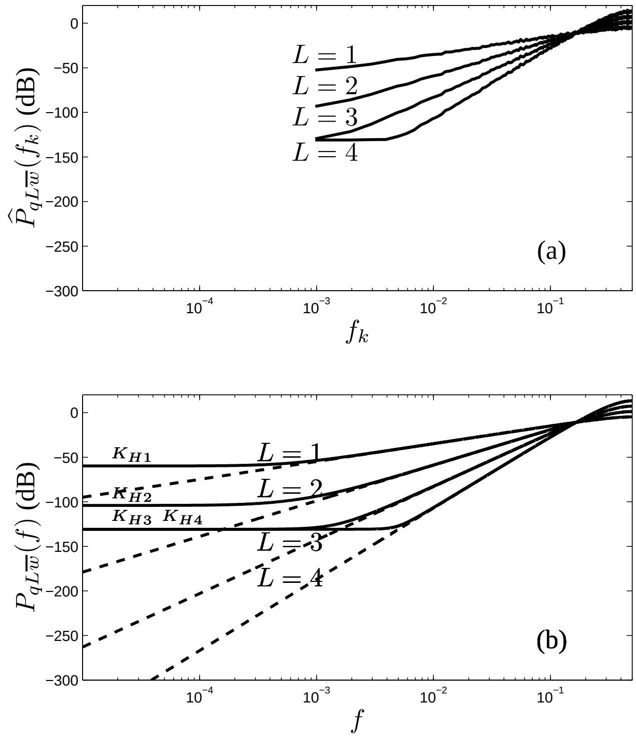

Figure 1. PSD of the output noise of a 1–bit $\Delta\Sigma$ modulator windowed by the Hanning sequence, estimated by means of the periodogram on 50 non–overlapped data records each of length $N = 2^{10}$ (a) and calculated by means of (2) (b). Bolded lines in each figure refer to four different order–shaping L as indicated by the corresponding labels. For comparison purposes, the magnitude of $P_L(f)$ has been graphed in (b) with dash–bolded lines.

For low frequency values, the PSD of each windowed noise converges to a constant value, K_{HL}, which can easily be calculated by means of (3)–(6) by considering that, for small values of f, $\cos(2\pi f) \simeq 1$, thus obtaining:

$$K_{H1} \simeq 2\sigma_e^2\frac{\pi^2}{N^2}, \tag{7}$$

$$K_{H2} \simeq \frac{16}{3}\sigma_e^2\frac{\pi^4}{N^4}, \tag{8}$$

$$K_{H3} = K_{H4} \simeq \frac{32}{3}\sigma_e^2\frac{\pi^4}{N^5}. \tag{9}$$

It results that such constants values decrease the higher N as shown in Fig. 2.

The frequency value f_{th_L} for which the theoretical PSD is equal to K_{HL}, can be calculated by solving the identity $K_{HL} = P_{qL}(f)$ for $L = 1, ..., 4$ and by considering that, for low frequency values, $\sin^{2L}(\pi f) \simeq (\pi f)^{2L}$, thus obtaining:

$$f_{th_1} \simeq \frac{1}{3^{1/2}}\frac{1}{N}, \tag{10}$$

$$f_{th_2} \simeq \frac{1}{3^{1/4}}\frac{1}{N}, \tag{11}$$

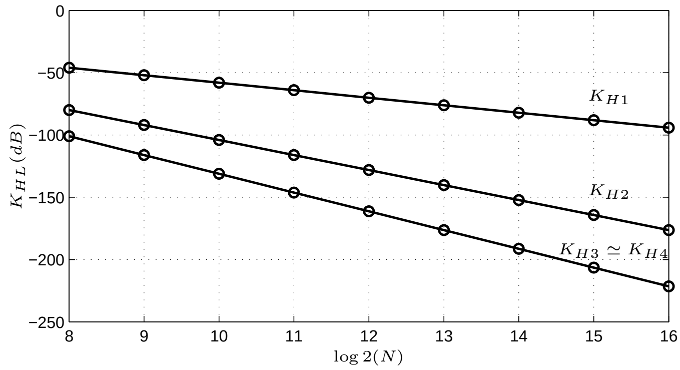

Figure 2. Constant values K_{HL}, expressed in dB, to which the PSD of the Hanning windowed noise converge in the low frequency region, as indicated by (7)–(9).

$$f_{th_3} \simeq \frac{1}{\pi^{1/3}}\frac{1}{6^{3/8}}\frac{1}{N^{5/6}}, \tag{12}$$

$$f_{th_4} \simeq \frac{1}{\pi^{1/2}}\frac{3^{1/4}}{6^{3/8}}\frac{1}{N^{5/8}}. \tag{13}$$

It should be noticed that for modulator orders lower than three the spectral leakage phenomenon does not significantly affect spectral estimation since, for any N, none of the periodogram bins lays in the frequency interval $[0, f_{th_L}[$. On the other hand, the number of bins included in that interval for the third and fourth order modulators depends on N, and is respectively equal to $0.35N^{1/6}$ and $0.38N^{3/8}$, thus increasing with N. It follows that the Hanning window sequence is a suitable two–term cosine window only for modulator orders lower than three.

By indicating with $\sigma^2_{q_L}$ and $\sigma^2_{q_{Lw}}$ the in–band powers of $q_L[\cdot]$ and $q_{Lw}[\cdot]$, respectively, the relative error of the shaped noise power estimation, $\varepsilon_{leak_L} \triangleq \left(\sigma^2_{q_{Lw}} - \sigma^2_{q_L}\right)/\sigma^2_{q_L}$, induced by the wide–band noise leakage phenomenon has been calculated as reported in App. B and it is approximately equal to:

$$\varepsilon_{leak_1} \simeq \frac{2}{N^2}OSR^2, \tag{14}$$

$$\varepsilon_{leak_2} \simeq \frac{40}{3}\frac{1}{N^4}OSR^4, \tag{15}$$

$$\varepsilon_{leak_3} \simeq \frac{112}{3}\frac{1}{\pi^4 N^5}OSR^6, \tag{16}$$

$$\varepsilon_{leak_4} \simeq 48\frac{1}{\pi^4 N^5}OSR^8. \tag{17}$$

It results that, for a given OSR, the higher the number of acquired samples, the lower the spectral leakage due to the wide–band component.

Expressions (14)–(17) have been used for evaluating the minimum N which bounds ε_{leak_L} within a given maximum value ε_{max}, thus obtaining:

$$N > OSR\left(\frac{2}{\varepsilon_{max}}\right)^{1/2}, \quad L = 1, \tag{18}$$

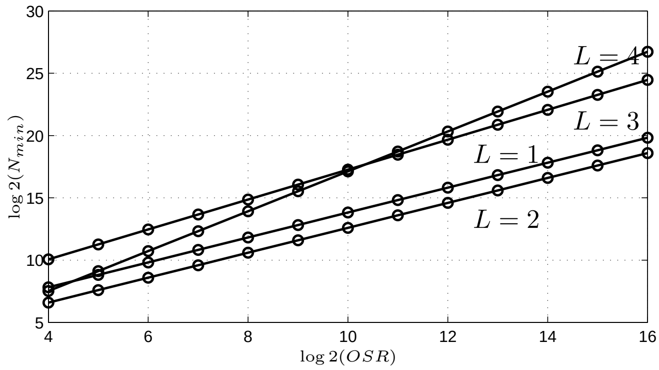

Figure 3. The minimum number of samples N which satisfies (18)–(21), as indicated by the corresponding labels, when $\varepsilon_{max} = 0.01$, versus OSR .

$$N > OSR\left(\frac{40}{3}\frac{1}{\varepsilon_{max}}\right)^{1/4}, \quad L = 2, \tag{19}$$

$$N > OSR^{6/5}\left(\frac{112}{3\pi^4}\frac{1}{\varepsilon_{max}}\right)^{1/5}, \quad L = 3, \tag{20}$$

$$N > OSR^{8/5}\left(\frac{48}{\pi^4}\frac{1}{\varepsilon_{max}}\right)^{1/5}, \quad L = 4. \tag{21}$$

Fig. 3 shows the two–base logarithm of the minimum N which satisfies (18)-(21) when $\varepsilon_{max} = 1\%$, versus the two–base logarithm of OSR. Such a figure can be used for evaluating the minimum number of samples to be employed for estimating the noise power of an L order modulator with a relative error lower than or equal to 0.01.

Thus, (18)–(21) can be employed for properly setting the parameters of the algorithm test used for estimating, with a given maximum relative error, the quantization noise power of a $\Delta\Sigma$ modulator with a specified OSR value.

III. CONCLUSIONS

Spectral leakage of the wide–band components affects the estimation of the in–band power of the $\Delta\Sigma$ shaped quantization noise, also when coherent sampling applies. In order to reduce such a phenomenon, output data are usually windowed by the Hanning sequence, which is the optimum two–term cosine window. In this paper, theoretical limits on the spectral estimation of the Hanning windowed shaped noise has been derived. In particular, the power spectral densities of the Hanning windowed modulators output has been derived for $\Delta\Sigma$ orders lower than 5, and it has been demonstrated that the Hanning sequence allows a good spectral estimation only for modulator orders lower than 3. The spectral leakage phenomenon modifies the low–frequency behavior of the PSD of higher order modulator. As a consequence, the quantization noise power estimation is affected by an error which increases with the used OSR and that can be reduced by employing a large number of acquired samples. In this paper, it has been indicated the criterion for choosing the minimum number of samples N for keeping bounded the relative in–band noise power error for each analyzed modulator order.

Appendix A

Derivation of expressions (2)–(6)

By indicating with $R_u[\cdot]$ the autocorrelation of the sequence $u[\cdot]$ and with $\mathcal{F}\{\cdot\}$ the Fourier Transform operator, the PSD of a windowed L–th order noise is:

$$\begin{aligned} P_{q_L w}(f) &= \mathcal{F}\{R_{q_L w}[m]\} = \\ &= \mathcal{F}\{R_{q_L}[m] R_{w_H}[m]\} = \\ &= P_{q_L}(f) + \mathcal{F}\{R_{q_L}[m]\left(R_{w_H}[m]-1\right)\} \end{aligned} \tag{A.1}$$

where $R_{w_H}[\cdot]$ is the aperiodic correlation sequence of the normalized Hanning window which can be written as [4]:

$$\begin{aligned} & R_{w_H}[m] = \\ = \; & \frac{2}{3} - \frac{2}{3N}|m| + \frac{1}{3N}\cos\left(\frac{2\pi}{N}|m|\right)(N-|m|) + \\ + \; & \frac{2}{3N}\left(\frac{\sin\left(\frac{2\pi}{N}\right)}{1-\cos\left(\frac{2\pi}{N}\right)} - \frac{\cos\left(\frac{2\pi}{N}\right)}{2\sin\left(\frac{2\pi}{N}\right)}\right)\sin\left(\frac{2\pi|m|}{N}\right), \\ & m = -(N-1), ..., (N-1) \end{aligned} \tag{A.2}$$

The autocorrelation of a white noise with zero–mean and variance equal to σ_e^2 filtered by an L–th order modulator with $L = 1, ..., 4$, $R_{q_L}[\cdot]$, can be easily calculated, thus obtaining a $2L+1$ length sequence centered in $m = 0$. By considering that $R_{w_H}[m] = R_{w_H}[-m]$ and that $R_{w_H}[0] = 1$, it results that the window correlation terms involved in the calculus of $R_{q_L w}[\cdot]$ are those defined for $m = 1, ..., L$. For such m values, the approximations $\cos(\alpha) \simeq 1 - \alpha^2/2 + \alpha^4/24$ and $\sin(\alpha) \simeq \alpha - \alpha^3/6 + \alpha^5/120$, with $\alpha \triangleq 2\pi m/OSR$, obtained by using a Taylor series expansion, hold true. By substituting such expressions in (A.2), the following results:

$$R_{w_H}[1] \simeq 1 - \frac{2}{3}\frac{\pi^2}{N^2} + \frac{2}{9}\frac{\pi^4}{N^4}, \tag{A.3}$$

$$R_{w_H}[2] \simeq 1 - \frac{8}{3}\frac{\pi^2}{N^2} + \frac{32}{9}\frac{\pi^4}{N^4} - \frac{8}{3}\frac{\pi^4}{N^5}, \tag{A.4}$$

$$R_{w_H}[3] \simeq 1 - 6\frac{\pi^2}{N^2} + 18\frac{\pi^4}{N^4} - \frac{64}{3}\frac{\pi^4}{N^5}, \tag{A.5}$$

$$R_{w_H}[4] \simeq 1 - \frac{32}{3}\frac{\pi^2}{N^2} + \frac{512}{9}\frac{\pi^4}{N^4} - \frac{272}{3}\frac{\pi^4}{N^5}. \tag{A.6}$$

By substituting $R_{q_L}[\cdot]$ for $L = 1, ..., 4$ and (A.3)– (A.6) in (A.1), (3)–(6) result.

Appendix B

Derivation of expressions (14)–(17)

$$\sigma_{q_L}^2 \triangleq 2\int_0^{1/2OSR} P_{q_L}(f)\mathrm{d}f \tag{B.1}$$

$$\begin{aligned} \sigma_{q_L w}^2 &\triangleq 2\int_0^{1/2OSR} P_{q_L w}(f)\mathrm{d}f = \\ &= 2\int_0^{1/2OSR} \left(P_{q_L}(f) + P_{q_L w}(f)\right)\mathrm{d}f. \end{aligned} \tag{B.2}$$

When high OSRs are employed, the approximations $\sin(2\pi f) \simeq 2\pi f$ and $\cos(2\pi f) \simeq 1$ holds true for frequency values lower than or equal to $1/2OSR$.

By substituting such expressions in $P_{q_L}(f)$ and $P_{q_L w}(f)$, and by considering that the relative error of the L–th order shaped noise power estimation is equal to $\varepsilon_{leak_L} \triangleq \left(\sigma_{q_L w}^2 - \sigma_{q_L}^2\right)/\sigma_{q_L}^2$, (14)–(17) result.

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