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COMPARISON BETWEEN DIFFERENT SOLUTIONS IN SPECTRUM ANALYSIS

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Abstract – In this paper the conventional equally spaced sampling technique for spectrum analysis is compared to an asynchronous random sampling strategy, which has been previously proposed by authors for other broadband digital instrumentation. To this aim, two new approaches to the performance analysis are exploited, both being characterised by the fact that no classical DFT algorithms are applied. In the first method, the analytical expression for the estimate of signal harmonic components is defined as a direct approximation of the Fourier series coefficients and the parameters that characterise the measurement accuracy associated with the considered sampling strategy are deduced. Following the second approach, the spectral component estimates are treated as random variables, due to the presence in their operative definition of unknown parameters that can be interpreted as stochastic. The expected value and variance are deduced for each harmonic estimate, in order to compare the properties of the two sampling strategies. Simulation results are proposed in order to validate the theoretical findings, showing an excellent agreement.

Keywords: spectrum analysis, equally spaced sampling, random sampling, performance analysis.

1. INTRODUCTION

It is well known that the digital spectral analysis of a periodic signal can be carried out by windowing the sampled signal and implicitly introducing its periodic repetition in order to obtain a discrete spectrum [1,2,3,4]. In this paper, the conventional equally spaced sampling technique for spectrum analysis is compared to a random asynchronous strategy, which has been introduced by the authors for a digital power meter [5,6], a vector voltmeter [7] and a power spectrum analyzer [8,9]. Two new different approaches are exploited to this aim: the first one allows to directly deduce the estimate of each spectral component by referring uniquely to the Fourier series, without the need for the application of the conventional Discrete Fourier Transform (DFT), and to easily evaluate the different parameters which characterize the spectrum analysis of a periodic signal as a function of the sampling strategy and the adopted window: bias, resolution, aliasing, leakage. Through this analysis the conditions in which it is convenient to adopt either the equally spaced or the random asynchronous sampling strategy can be easily deduced.

Since in both equally spaced and random sampling strategies the analytical expression of the harmonic estimates depends on unknown parameters, another method has been introduced by the authors, which treats such parameters as random variables statistically characterised, following the Bayesan approach [9,10], on the basis of *a priori* information. The statistical parameters associated with the spectral component estimates, capable of quantifying the properties of each sampling strategy, are deduced and evaluated. Simulation results are then presented in order to validate the theoretical findings.

2. DIRECT APPROACH TO SPECTRUM ANALYSIS

We consider the problem of measuring the spectral components X_q of a periodic signal $x(t)$ having a finite spectrum:

$$x(t) = \sum_{q=-M}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi q f_1 t} \tag{1}$$

where $f_1 = 1/T_1$ is the fundamental frequency and M the practically finite maximum order of harmonics. According to the Fourier series theory, the unknown spectral components can be expressed in the form:

$$X_n = \frac{1}{T_1} \int_{-T_1/2}^{+T_1/2} x(t) \mathrm{e}^{-\mathrm{j}2\pi n f_1 t} \mathrm{d}t \quad \text{with } -M \le n \le +M \tag{2}$$

where $X_{-n} = X_n^*$ if $x(t)$ is real. In the digital implementation of a spectrum analyzer the integral in (2) must be necessarily approximated by a finite sum of signal sampled values each multiplied by the correspondent discrete periodic exponential signal:

$$\hat{X}_n(p_0, N) = \frac{1}{2N+1} \sum_{i=p_0-N}^{p_0+N} x(t_i) \mathrm{e}^{-\mathrm{j}2\pi n f_1 t_i} \tag{3}$$

where p_0 is a positive integer which marks the centred value of the sequence of $2N+1$ samples and t_i indicates a generic sampling instant. Equation 3 represents an estimate of the n^{th} spectral component due both to an observation interval independent from the period of the input signal and to a finite sum of sampled values of it. On the hypothesis of an equally spaced sampling strategy it results:

$$t_i = iT_s + \tau \tag{4}$$

where τ is the unknown delay between the signal time origin and the sampling time generator, and T_s represents the sampling rate; instead, when using the random sampling strategy previously introduced by the authors [5] the sampling instant is defined as follows:

$$t_i = (i + Y_i)T_s + \tau \tag{5}$$

where now T_s represents an "average" sampling rate while Y_i is a set of independent random variables whose probability density function is uniformly distributed between $-1/2$ and $+1/2$.

By substituting (4) into (3), we obtain the estimate $\hat{X}_n$ in the hypothesis of an equally spaced sampling technique (which we distinguish through the pedix *ES*):

$$\begin{aligned}\hat{X}_{nES}(\tau, p_0, N) = \hat{X}_{nES} &= \sum_{i=p_0-N}^{p_0+N} \sum_{q=-M}^{+M} X_q e^{j2\pi(q-n)f_1(\tau+iT_s)} = \\ &= X_n + \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q e^{j2\pi(q-n)f_1(\tau+p_0T_s)} W_{ES}\left((2N+1)(q-n)f_1T_s\right)\end{aligned} \tag{6}$$

In fact it results:

$$\begin{aligned}\frac{1}{2N+1}\sum_{i=p_0-N}^{p_0+N} e^{j2\pi(q-n)if_1T_s} &= \\ &= e^{j2\pi(q-n)p_0f_1T_s}\frac{\mathrm{sinc}\left[(2N+1)(q-n)f_1T_s\right]}{\mathrm{sinc}\left[(q-n)f_1T_s\right]}\end{aligned} \tag{7}$$

By defining the function:

$$W_{ES}\left((2N+1)fT_s\right) = \frac{\mathrm{sinc}\left[(2N+1)fT_s\right]}{\mathrm{sinc}(fT_s)} \tag{8}$$

with $f = (q-n)f_1$, and substituting (7) into the second expression of (6), we obtain the last expression of (6). It is interesting to observe that the function $W_{ES}\left((2N+1)fT_s\right)$ coincides with the Fourier Transform of the discrete rectangular window of $2N+1$ values and it is known as Dirichlet kernel [2,10]. It is periodic of period $fT_s = 1$ (i.e. $f = f_s$) with a maximum value equal to one for fT_s integer and, within the first period beginning from zero, it is symmetric with respect to $fT_s = 0.5$ (i.e. $f = f_s/2$) and it is null for $(2N+1)fT_s = k$ (i.e. $f = k\frac{f_s}{2N+1}$), with k an integer less than $2N+1$; for $fT_s = 0.5$ the absolute value of the function is equal to $1/(2N+1)$. It is characterized by equal main lobes, around the integer values of fT_s, and side-lobes between the successive zeroes. The last but one expression of (6) shows that the estimate of each spectral component of a periodic signal sampled with a window can be obtained as the sum of the contributions of all the original spectral components each one multiplied by the correspondent value of the Fourier Transform of the discrete window centred on the considered spectral component (for example $-Mf_1$ in fig.1). In order to separate adequately the contributions of spectral components contiguous to the considered one, i.e. to obtain an adequate resolution, it is necessary to impose a minimum distance between the spectral components of the original signal, i.e. f_1, greater than the half-width of the main lobe of the Fourier transform of the discrete window, i.e. $f_1 > f_s/(2N+1)$. In other terms, the observation interval $(2N+1)T_s$, i.e. the width of the window, must be greater than the period T_1 of the input signal. Further, due to the periodicity of the function $W_{ES}\left((2N+1)fT_s\right)$, it is necessary to avoid that the main lobes nearer to that one centred on the estimated spectral component affect the contributions of the most far spectral components of the signal; this phenomenon is known as aliasing. To this end it is necessary to consider the Fourier transform of the window centred on the spectral component $-Mf_1$ (or, analogously, $+Mf_1$) and verify that the successive (or, analogously, previous) main lobe does not involve the most far spectral component Mf_1 (or, analogously, $-Mf_1$), as it is shown in fig.1. Because the period of the function W_{ES} is $f_s = 1/T_s$ and the half width of the main lobe is equal to $f_s/(2N+1)$, it is therefore necessary to impose $2Mf_1 < f_s\left(1 - \frac{1}{2N+1}\right) = \frac{2N}{2N+1}f_s$. The two conditions, relative to resolution and aliasing, have the overall effect to impose $N > M$, i.e. the number of samples must be greater than the number of spectral components of the input signal. Finally, the second term into the last expression of(6) represents the contribution of the other components different from the estimated one, phenomenon which is called

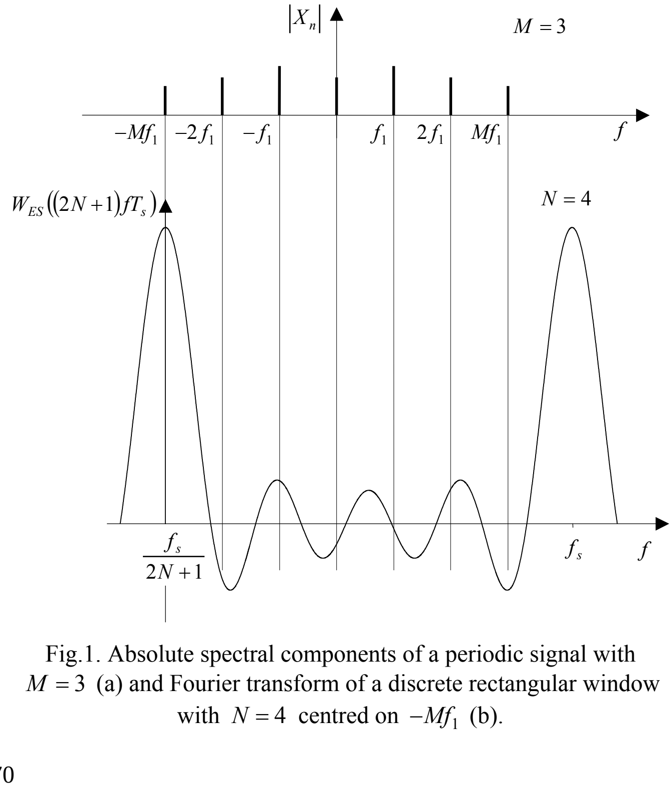

Fig.1. Absolute spectral components of a periodic signal with $M = 3$ (a) and Fourier transform of a discrete rectangular window with $N = 4$ centred on $-Mf_1$ (b).

leakage; theoretically it should be null, but it is different from zero due to the presence of the side-lobes of the function $W_{ES}\left[(2N+1)fT_s\right]$ whose absolute maximum amplitude can be considered with acceptable approximation inversely proportional to the number of samples of the window, $2N+1$. The preceding treatment can be interpreted as the sampling theorem relative to a periodic signal sampled when a discrete rectangular window is used; it is important to observe that, also if the sampling theorem is satisfied, the original signal cannot be reconstructed due to the presence of the leakage. In fact, by substituting (6) into (1), the estimate $\hat{x}_{ES}(t)$ of the input signal results:

$$\hat{x}_{ES}(t) = \sum_{n=-M}^{+M} \hat{X}_{nES} \mathrm{e}^{\mathrm{j}2\pi q f_1 t} = x(t) + \sum_{n=-M}^{+M} \mathrm{e}^{\mathrm{j}2\pi n f_1 t} \cdot \\ \cdot \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{e}^{-\mathrm{j}2\pi(q-n) f_1 (\tau + p_0 T_s)} W_{ES}\left((2N+1)(q-n) f_1 T_s\right) \tag{9}$$

When the condition $f_1 = b f_s /(2N+1)$, with b an integer less than $2N+1$, is satisfied, i.e. the sampling is synchronous with the input signal because the $(2N+1)$ samples are taken in b periods of the input signal, the function $W_{ES}\left[(2N+1)(q-n)f_1T_s\right] = W_{ES}\left[(q-n)b\right]$ is always null except for $q-n = 2N+1$ where it assumes a value equal to one (see (8)). In order to avoid this situation, which is known as the aliasing effect, since the maximum value of $q-n$ is $2M$, we must impose that $2M < 2N+1$, i.e. $N \geq M$. In conclusion, when both the conditions $f_1 = b f_s /(2N+1)$ and $N \geq M$ are satisfied, the problem of resolution is automatically overcome and the leakage is null; in fact the positions of the spectral components different from the estimated one coincide with the zeroes of the lobes of the Dirichlet kernel and the second term in (9) becomes null. Therefore the estimates $\hat{x}_{ES}(t)$ coincides with the original signal.

When the sampling instant is instead random, defined according to (5), by substituting this equation into (3) the estimator $\hat{X}_n$, which is distinguished through the pedix *R*, becomes:

$$\hat{X}_{nR}(\tau, p_0, N, \underline{Y}) = \hat{X}_{nR} = \\ = X_n + \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n) f_1 \tau} \frac{1}{2N+1} \sum_{i=p_0-N}^{p_0+N} \mathrm{e}^{\mathrm{j}2\pi(q-n) i f_1 T_s} \mathrm{e}^{\mathrm{j}2\pi(q-n) Y_i f_1 T_s} \tag{10}$$

Due to the vector of the random variables Y_i the quantity $\hat{X}_{nR}$ is a random variable itself; therefore, it can be characterized by its expected value with respect to the vector of the random variables Y_i and therefore the pedix Y is introduced:

$$E_Y\left\{\hat{X}_{nR}\right\} = \\ = X_n + \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n) f_1 (\tau + p_0 T_s)} W_R\left((2N+1)(q-n) f_1 T_s\right) \tag{11}$$

In fact, by recalling (7), taking into account that:

$$\mathrm{E}\left\{\mathrm{e}^{\mathrm{j}2\pi(q-n) Y_i f_1 T_s}\right\} = \int_{-1/2}^{+1/2} \mathrm{e}^{\mathrm{j}2\pi(q-n) y f_1 T_s} \mathrm{d}y = \mathrm{sinc}\left[(q-n) f_1 T_s\right] \tag{12}$$

we obtain the two last expressions of (11), where a weighting function W_R has been defined as

$$W_R\left((2N+1)fT_s\right) = sinc\left[(2N+1)fT_s\right] \tag{13}$$

and evaluated for $f = (q-n)f_1$. The function $W_R\left((2N+1)fT_s\right)$ coincides with the Fourier Transform of the continuous rectangular window of width $(2N+1)T_s$; it is characterized by a main lobe for $fT_s = 0$ of width $f = 2f_s/(2N+1)$ and successive side-lobes of decreasing amplitudes. The last but one expression of (11) shows that the expected value of each spectral component of a periodic signal randomly sampled and windowed can be obtained as the sum of the contributions of all the original spectral components each one multiplied by the correspondent value of the Fourier Transform of the continuous rectangular window centred on the estimated spectral component (for example $-Mf_1$ in fig.2). Also in this case, in order to separate adequately the contributions of spectral components contiguous to the considered one, i.e. to obtain an adequate resolution, it is necessary to impose the condition $f_1 > f_s/(2N+1)$. In other terms, the observation interval $(2N+1)T_s$, i.e. the width of the window, must be greater than the period T_1 of the input signal. Because the function $W_R\left((2N+1)fT_s\right)$ is not periodic, the aliasing effect in this case does not exist unlike the equally spaced sampling strategy. Therefore the condition relative to the resolution must be satisfied independently by the number of spectral components of the input signal. The leakage effects, due to the sidelobes of the window, is presents also in this sampling strategy; however its

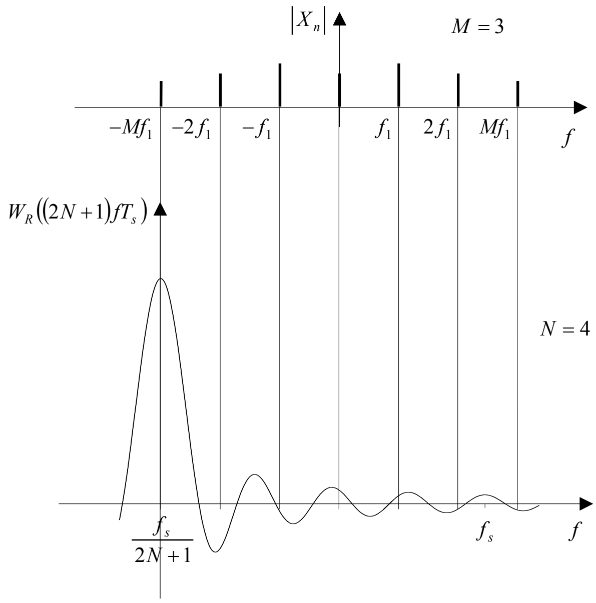

Fig. 2. Absolute spectral components of a periodic signal with $M = 3$ (a) and Fourier transform of a continuous rectangular window with $N = 3$ centred on $-Mf_1$ (b).

effect is decreasing moving away from the main lobe. Therefore also in this case the original signal cannot be reconstructed due to the leakage effect. In order to take into account the variability of the estimator, by referring to (10) we can introduce the mean square error defined as follows:

$$\begin{aligned} &\mathrm{E}\left\{\left|\hat{X}_{nR} - X_n\right|^2\right\} = \frac{1}{(2N+1)^2} \cdot \\ &\cdot \mathrm{E}\left\{\left|\sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \sum_{i=p_0-N}^{p_0+N} \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(\tau+iT_s+Y_iT_s)}\right|^2\right\} \le \frac{1}{2N+1}\sum_{\substack{q=-M \\ q\neq n}}^{+M}\left|X_q\right|^2 \end{aligned} \tag{14}$$

The last passage is due to the well known property that the square modulus of the sum is not greater than the sum of the square modula of the addenda. This inequality shows that the maximum mean square error is inversely proportional to $1/(2N+1)$. In order to compare this result with the analogous expression for the equally-spaced sampling strategy, we can observe that it results:

$$\begin{aligned} &\left|\hat{X}_{nES} - X_n\right|^2 \cong \\ &\cong \frac{1}{(2N+1)^2}\left|\sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1(\tau+p_0T_s)}\right|^2 \le \frac{1}{(2N+1)^2}\sum_{\substack{q=-M \\ q\neq n}}^{+M}\left|X_q\right|^2 \end{aligned} \tag{15}$$

In the last passage we have supposed negligible the aliasing effect and we have assumed constant the function $W_{ES}((2N+1)fT_s)$, nearly approximated by its central value $1/(2N+1)$: in this case the mean square error is inversely proportional to $1/(2N+1)^2$ while in the random previous case it was inversely proportional to $1/(2N+1)$. Therefore the proposed random strategy is characterized by the lack of aliasing, but its mean square error is greater under the same width of the window. In order to obtain the same mean square error, a longer observation interval is then needed.

3. STATISTICAL APPROACH TO SPECTRUM ANALYSIS

Since the estimate $\hat{X}_{nR}$ of (10) is a complex random variable, its mean value $\mathrm{E}\{\hat{X}_{nR}\}$ is given by:

$$\mathrm{E}\{\hat{X}_{nR}\} = \mathrm{E}\{\mathrm{Re}[\hat{X}_{nR}]\} + \mathrm{j}\mathrm{E}\{\mathrm{Im}[\hat{X}_{nR}]\} \tag{16}$$

due to the property of a linear operator; its mean square error, indeed, results:

$$\begin{aligned} &\mathrm{E}\{(\hat{X}_{nR} - X_n)(\hat{X}^*_{nR} - X^*_n)\} = \\ &= [X_n - \mathrm{E}\{\hat{X}_{nR}\}][X^*_n - \mathrm{E}\{\hat{X}^*_{nR}\}] + \mathrm{Var}\{\hat{X}_{nR}\} \end{aligned} \tag{17}$$

where we have introduced the variance defined as follows:

$$\begin{aligned} &\mathrm{Var}\{\hat{X}_{nR}\} = \mathrm{E}\{[\hat{X}_{nR} - \mathrm{E}\{\hat{X}_{nR}\}][\hat{X}^*_{nR} - \mathrm{E}\{\hat{X}^*_{nR}\}]\} = \\ &= \mathrm{Var}\{\mathrm{Re}[\hat{X}_{nR}]\} + \mathrm{Var}\{\mathrm{Im}[\hat{X}_{nR}]\} \end{aligned} \tag{18}$$

Therefore the mean square error depends both on the difference of the mean value with respect to the theoretical value (i.e. the bias) and the variability of the estimated value with respect to its mean value. By considering (10) we can deduce:

$$\mathrm{Var}_Y\{\hat{X}_{nR}\} \cong \frac{1}{2N+1}\sum_{q=-M}^{+M} X_q X^*_q\left[1 - \mathrm{sinc}^2[(q-n)f_1T_s]\right] \tag{19}$$

where in the last passage we have supposed that $(2N+1)f_1T_s >> 1$ so that the correspondent *sinc* function is practically null for $q \neq q'$; therefore only the spectral components for $q = q'$ must be taken into account.

It is interesting to observe that, both in (6) and (10), the estimates of the spectral component are conditioned to a particular value of the delay τ and to the centred value p_0 of the sequence of $2N+1$ samples [10,11]. The unknown delay τ between time origins of the signal and sampling generators can be interpreted as a continuous random variable, independent of the vector of the random variables $\{Y_i\}$ and uniformly distributed into a generic time interval $-T'/2,+T'/2$. In the same way, the unknown centred value p_0 of the considered sequence can be interpreted as a discrete random variable uniformly distribute in a generic discrete interval $(-m,+m)$. In this hypothesis also the estimate given by (6) becomes a random variable and its expected value results:

$$\begin{aligned} &\mathrm{E}\{\hat{X}_{nES}\} = \\ &= X_n + \sum_{\substack{q=-M \\ q\neq n}}^{+M} X_q \mathrm{sinc}[(q-n)f_1T'] \frac{\mathrm{sinc}[(q-n)f_1(2m+1)T_s]}{\mathrm{sinc}[(q-n)f_1T_s]} \cdot \\ &\cdot W_{ES}((2N+1)(q-n)f_1T_s) \end{aligned} \tag{20}$$

since in analogy to (12) and (7) we respectively have:

$$\mathrm{E}\{\mathrm{e}^{\mathrm{j}2\pi(q-n)f_1\tau}\} = \frac{1}{T'}\int_{-T'/2}^{+T'/2} \mathrm{e}^{\mathrm{j}2\pi(q-n)f_1\tau}\mathrm{d}\tau = \mathrm{sinc}[(q-n)f_1T'] \tag{21}$$

$$\mathrm{E}\{\mathrm{e}^{-\mathrm{j}2\pi(q-n)f_1p_0T_s}\} = \frac{\mathrm{sinc}[(2m+1)(q-n)f_1T_s]}{\mathrm{sinc}[(q-n)f_1T_s]} \tag{22}$$

In order to obtain a result independent of the measurement occasion, it is necessary to consider the asymptotic expected value of $\mathrm{E}\{\hat{X}_{nES}\}$, i.e. for $T' \to \infty$ and/or $m \to \infty$. From (20) we deduce that this asymptotic expected value coincides with X_n and it represents an unbiased estimator of the n^{th} spectral component:

$$\underset{\to}{\mathrm{E}}\{\hat{X}_{nES}\} = X_n \tag{23}$$

Therefore it is convenient to consider, instead of a single measurement result, where the leakage depends on the unknown delay τ and on the centred value p_0, the mean of different measurements. Because the asymptotic bias is null, the asymptotic mean square error coincides with the asymptotic variance. It can be shown that this asymptotic variance results [10,11]:

$$\underset{\rightarrow}{\operatorname{Var}}\left\{\hat{X}_{nES}\right\}=\sum_{\substack{q=-M\\ q\neq n}}^{+M} X_q X_q^* H_{ES}\left[(2N+1)(q-n)f_1 T_s\right] \quad (24)$$

where we have introduced the weighting function $H_{ES}((2N+1)fT_s)$ which coincides, in this case, with the squared Fourier Transform of the discrete rectangular window:

$$H_{ES}((2N+1)fT_s)=W_{ES}^2((2N+1)fT_s) \quad (25)$$

Also this weighting function $H_{ES}[(2N+1)fT_s]$ is periodic of period f_s and, within each period, assumes $2N$ zeroes. The comments to (6) can now be repeated for the variance given by (24) and the results are substantially the same [10]. In particular the maximum amplitude of the central sidelobe is $1/(2N+1)^2$; therefore the contribution to the mean square error of each spectral component depends on the squared amplitude of each spectral component multiplied for a quantity which in the worst case can be assumed approximately equal to $1/(2N+1)^2$. This is the result previously obtained with (15).

In analogy to the previous procedure, taking into account (21) and (22) in the case of the random strategy defined by (5), from (11) we obtain:

$$\begin{aligned}\mathrm{E}\left\{\mathrm{E}_Y\left\{\hat{X}_{nR}\right\}\right\}&=\mathrm{E}\left\{\hat{X}_{nR}\right\}=X_n+\sum_{\substack{q=-M\\ q\neq n}}^{+M} X_q \operatorname{sinc}\left[(q-n)f_1 T'\right]\cdot\\ &\cdot\frac{\operatorname{sinc}\left[(q-n)f_1(2m+1)T_s\right]}{\operatorname{sinc}\left[(q-n)f_1 T_s\right]} W_R((2N+1)(q-n)f_1 T_s)\end{aligned} \quad (26)$$

Also in this case the asymptotic expected value coincides with X_n and it is an unbiased estimate of the n^{th} spectral component:

$$\underset{\rightarrow}{\mathrm{E}}\left\{\hat{X}_{nR}\right\}=X_n \quad (27)$$

Therefore the asymptotic mean square error is due uniquely to the variance. Because analogously to (27) it results $\underset{\rightarrow}{\mathrm{E}}\left\{\hat{X}_{nR}^*\right\}=X_n^*$, we can deduce:

$$\underset{\rightarrow}{\operatorname{Var}}\left\{\hat{X}_{nR}\right\}=\underset{\rightarrow}{\mathrm{E}}\left\{\hat{X}_{nR}\hat{X}_{nR}^*\right\}-X_n X_n^* \quad (28)$$

It can also be shown that we can write:

$$\begin{aligned}&\mathrm{E}\left\{\mathrm{E}_Y\left\{\hat{X}_{nR}\hat{X}_{nR}^*\right\}\right\}=\mathrm{E}\left\{\hat{X}_{nR}\hat{X}_{nR}^*\right\}=\\ &=\frac{1}{2N+1}\sum_{q=-M}^{+M}\sum_{q'=-M}^{+M} X_q X_{q'}^* \mathrm{E}\left\{e^{j2\pi(q-q')f_1\tau}\right\}\mathrm{E}\left\{e^{j2\pi(q-q')f_1 p_0 T_s}\right\}\cdot\\ &\cdot\Big\{\operatorname{sinc}\left[(2N+1)(q-q')f_1 T_s\right]+\\ &+(2N+1)\operatorname{sinc}\left[(2N+1)(q-n)f_1 T_s\right]\operatorname{sinc}\left[(2N+1)(q'-n)f_1 T_s\right]\\ &-\operatorname{sinc}\left[(q-n)f_1 T_s\right]\operatorname{sinc}\left[(q'-n)f_1 T_s\right]\frac{\operatorname{sinc}\left[(2N+1)(q-q')f_1 T_s\right]}{\operatorname{sinc}\left[(q-q')f_1 T_s\right]}\Big\}\end{aligned} \quad (29)$$

because τ and p_0 are independent. In analogy to (21) and (22) we obtain:

$$\begin{aligned}&\mathrm{E}\left\{\hat{X}_{nR}\hat{X}_{nR}^*\right\}=\frac{1}{2N+1}\sum_{q=-M}^{+M}\sum_{q'=-M}^{+M} X_q X_{q'}^* \operatorname{sinc}\left[(q-q')f_1 T'\right]\cdot\\ &\cdot\frac{\operatorname{sinc}\left[(q-q')f_1(2m+1)T_s\right]}{\operatorname{sinc}\left[(q-q')f_1 T_s\right]}\Big\{\operatorname{sinc}\left[(2N+1)(q-q')f_1 T_s\right]+\\ &+(2N+1)\operatorname{sinc}\left[(2N+1)(q-n)f_1 T_s\right]\operatorname{sinc}\left[(2N+1)(q'-n)f_1 T_s\right]-\\ &-\operatorname{sinc}\left[(q-n)f_1 T_s\right]\operatorname{sinc}\left[(q'-n)f_1 T_s\right]\frac{\operatorname{sinc}\left[(2N+1)(q-q')f_1 T_s\right]}{\operatorname{sinc}\left[(q-q')f_1 T_s\right]}\Big\}\end{aligned} \quad (30)$$

When we consider the asymptotic value with respect to T' or m, all the terms with $q\neq q'$ becomes null and therefore it results:

$$\begin{aligned}&\underset{\rightarrow}{\mathrm{E}}\left\{\hat{X}_{nR}\hat{X}_{nR}^*\right\}=\sum_{q=-M}^{+M} X_q X_{q'}^*\cdot\\ &\cdot\left\{\frac{1}{2N+1}\left(1-\operatorname{sinc}^2\left[(q-n)f_1 T_s\right]\right)+\operatorname{sinc}^2\left[(2N+1)(q-n)f_1 T_s\right]\right\}\end{aligned} \quad (31)$$

By substituting this expression into (27), by separating the contribution for $q=n$ and by recalling (13) we obtain:

$$\underset{\rightarrow}{\operatorname{Var}}\left\{\hat{X}_{nR}\right\}=\sum_{\substack{q=-M\\ q\neq n}}^{+M} X_q X_q^* H_R\left[(2N+1)(q-n)f_1 T_s\right] \quad (32)$$

where the weighting function H_R for the proposed random strategy is defined as follows:

$$\begin{aligned}&H_R\left[(2N+1)fT_s\right]=\\ &=\left[\frac{1}{2N+1}\left(1-\operatorname{sinc}^2\left[fT_s\right]\right)+\operatorname{sinc}^2\left[(2N+1)fT_s\right]\right]\end{aligned} \quad (33)$$

with $f=(q-n)f_1$ and coincides with that deduced previously for the random sampling wattmeter [5]. For $f=0$ it results $H_R(0)=1$; for $f\cong 0$ it results $W_R^2[fT_s]\cong 1$; therefore we obtain $H_R[(2N+1)fT_s]\cong W_R^2[(2N+1)fT_s]$. Successively we approximately have $\frac{W_R^2[fT_s]}{2N+1}\cong W_R^2[(2N+1)fT_s]$; therefore the weighting function becomes a constant, function of the number of samples of the window: $H_R[(2N+1)fT_s]\cong\frac{1}{2N+1}$. Figure 3 shows the shape of the weighting function $H_R[(2N+1)fT_s]$ as a function of fT_s and $(2N+1)$. From this figure we can conclude that the proposed random sampling does not introduce any bandwidth limitation; therefore the bandwidth of the instrument is limited uniquely by the S/H circuits adopted. It is important to observe, in comparison with the equally spaced sampling strategies, that the contribution to the mean square error of each spectral component can be deduced multiplying the squared amplitude of the spectral component by $1/(2N+1)$ in the proposed random sampling technique and by a quantity which in the worst case can be assumed equal to $1/(2N+1)^2$ in the equally spaced sampling strategy. Therefore this random technique, which is not limited in bandwidth, requires a greater length of the window in order to adequately reduce the leakage.

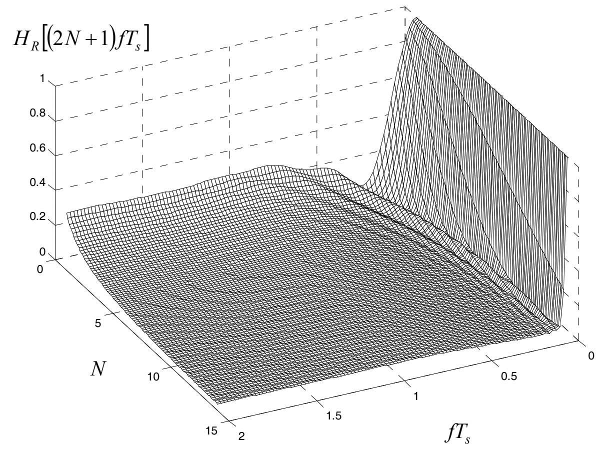

Fig. 3. Shape of the weighting function $H_R[(2N+1)fT_s]$ for the proposed random sampling strategy as a function with respect to both fT_s and *N*.

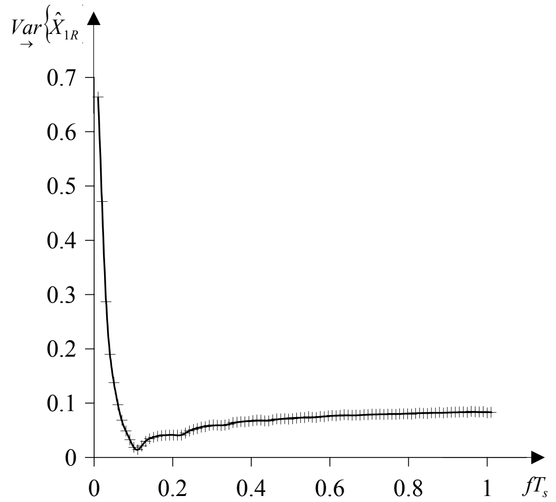

Fig. 4. Comparison between the theoretical (continuous line) and simulated (+) results of the asymptotic variance (32) relative to the fundamental spectral component on the hypothesis of a random sampling strategy defined according to (5) with $T_s = 1.01$ ms and a signal with two spectral components.

4. SIMULATED RESULTS

By considering $N = 4$ successive samples of the sum of a sinusoidal and a cosinusoidal signals of unit amplitude respectively at frequency f_1 and $2f_1$, by assuming a random-sampling strategy defined according to (5) with $T_s = 1.01$ ms, the asymptotic variance of the fundamental (fig.4) was evaluated by considering 10^4 successive measurements. The simulated results are practically coincident with the theoretical ones and confirm that this sampling strategy does not introduce any bandwidth limitation.

6. CONCLUSIONS

A new procedure, which allows to quantify all the errors associated with the estimate of the spectral components of a periodic signal through a finite number of its equally spaced samples has been introduced, by referring uniquely to the formulae of the Fourier series. It has been shown that the number of samples in the time domain of the input signal and the sampling frequency must be fixed in order to avoid the aliasing effect and both to obtain an adequate resolution and to reduce the leakage effect. The number of values of the frequency spectrum associated to the finite sequence of the input signal must instead be selected on the basis of the amplitude and phase error which can be accepted.

The same procedure was applied to compare the equally spaced sampling with a random one, in which the samples are randomly selected, with uniform distribution, in equal successive time intervals. It was shown that the aliasing effect is not present in this random sampling; however to obtain the same leakage effect of the equally spaced sampling a greater number of samples must be selected.

Successively, these two sampling techniques were studied by introducing a different approach, which is based on the hypothesis of randomly selecting, on the basis of the Bayesan approach, all the unknown parameters. The corresponding asymptotic statistical parameters show that these sampling strategies are unbiased and that the asymptotic variance relative to each spectral component depends also on the other spectral components weighted by a proper function. Since such a weighting function is different for each sampling strategy, it can be assumed as a parameter for its characterisation. The simulated results confirm the validity of this assumption.

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