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FROM GENERIC TO SYNTHETIC GRAVITY MODELING - A COMPARISON FROM METROLOGIC PERSPECTIVE

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Abstract − Many mechanical quantities in metrology are affected by the local gravity. Modern gravimetric methods allow to determine the local gravity value with sufficient accuracy, but the effort is still time consuming and expensive. An interpolation within available gravity databases mostly does not suffice because of an inadequate data density compared with the heterogeneous field. This study gives a short overview of today's most important representations of the physical gravity field. The classical methods will be introduced from the perspective of the usability in metrology and compared to some strategies in synthetic gravity modeling.

Keywords: local gravity, synthetic gravity, digital terrain model

1. INTRODUCTION

The local gravity value plays an important role in precision measurements of many mechanical quantities, i.e. the force of gravity, the mass, the pressure or even the electrical intensity. Newton's law of gravitation serves as the basis for numerous applications in physics or especially in metrology. According to the particular purpose the applications are connected to relative uncertainties in a range from 10^{-2} to 10^{-6} or even less (see Fig.1). However, the local gravity is assumed as known within the same, respective less uncertainty.

This can be covered with modern gravimetric methods easily which are presently limited three orders of magnitude beneath, at 10^{-9} relative.

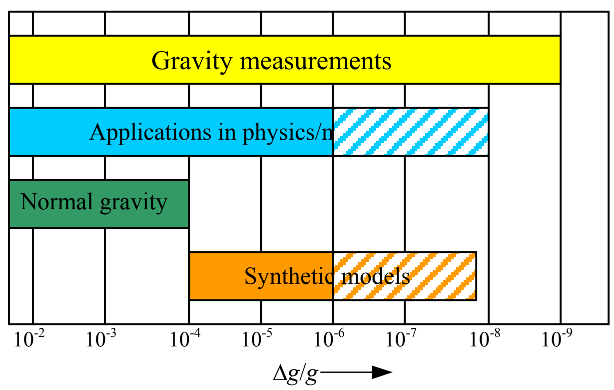

Fig. 1. Relative uncertainties of methods to provide gravity parameters and applications affected by gravity

In practical sense those gravity measurements are both expensive and very time consuming in relation to a required point density.

Furthermore, a connection to a local reference network can be necessary depending on the type of measurement. Thus, existing databases of recent measurements normally do not have a sufficient density of gravity values to provide a required uncertainty for each place.

One reason for this can be found in the inhomogeneous gravity field due to a irregular distribution of mass in the interior of the earth. A simple approximation of this field has been recommended by the International Association of Geodesy (IAG) in 1979 [1]. Because of its definition via a geodetic reference system it serves as a 'Normal' gravity field. The differences to the physical field are described as gravity anomalies which can reach a magnitude of 10^{-4} relative. This corresponds to the accuracy level of the normal gravity.

Applications of higher resolutions require a refinement of the global model based on the development of generic and synthetic gravity models (see Fig.1). The latter break a new ground by the avoidance of gravity data as the basis of computation. With the help of external data sources, like terrain models, geological structures and density models, gravity modeling may achieve the requisite accuracy to become independent from national gravity databases.

Future activities in this regard will be associated with the object to connect metrologic procedures referring to force, mass and torque to the gravity field via synthetic models. Present models in contrast do not provide gravity at 10^{-6} relative which is required in the according error budgets.

2. GENERIC REPRESENTATIONS OF THE GRAVITY FIELD

The attractive force F after Newton's Law can easily be transduced into a gravitational acceleration b by setting the attracted point mass to unity. For practical computations in the gravity field the vector quantity b can be simplified by the scalar quantity of the potential V. The relation may be given as

$$b = \text{grad } V \qquad (0)$$

respective modifcated to gravity

$$g = \text{grad}\, W\,. \qquad (2)$$

An analytical description of the gravity potential only succeeds at a bounding surface, under which all masses are concentrated. For points beneath this surface a function of V does not correspond to Laplace's differential equation of second order

$$\Delta V_{Interior} = \text{div grad}\, V_{Interior} = 0\,. \qquad (3)$$

Basically this results from several singularities of the density distribution in the spherical layered earth structure, which is not known with sufficient accuracy. For the same reason a gravity model at present can only be derived indirectly from observations above the surface and hypotheses of the interior.

2.1. Normal gravity field

Assuming the singularities are neglected and a homogeneous structure is adopted, the main part of the gravity field can be approximated . Moritz [2] has given an analytical approach for the gravity potential at an equipotential surface U_0 by introducing four conventional constants of the Earth body. Its shape is fitted by an rational symmetrical ellipsoid that is located in the Earth center defined by the semi-major axis a and a dynamical form factor J_2. The physical properties are described by the gravitational mass effect, set by the geocentric gravitational constant *GM*, and the Earth rotation, set by the angular velocity ω. The so-called *Geodetic Reference System 1980* combines therewith the postulations of a manageable geometry for the simplification of mathematical transformations and a normal gravity field as a reference for high-resolution gravity field representations.

In the exterior space the potential U_0 can be developed in spherical harmonics (*see* 1.3) that serve as a basis for the determination of further coefficients.

According to eq. (1) the gravity on the ellipsoid surface can be derived from U_0 as a function of the place. Due to the definition of U_0 the *normal gravity* γ_0 can be represented in a closed formula but for practical use the following series expansion is easier to handle:

$$\begin{aligned}\gamma_0(\varphi) = 9.780327 \cdot (1 &+ 0.0052790414 \cdot \sin^2\varphi \\ &+ 0.0000232718 \cdot \sin^4\varphi \\ &+ 0.0000001262 \cdot \sin^6\varphi \\ &+ 0.0000000007 \cdot \sin^8\varphi)\ \mathrm{m/s^2}\end{aligned} \qquad (4)$$

with φ : latitude

Comparing to the closed form, eq. (4) reaches a relative uncertainty of 10^{-10}. For applications with less demands eq. (4) can easily be truncated after the 4^{th} or 6^{th} order.

In the exterior of the reference surface the change of gravity with height h can be treated approximately as linear. To cover world's entire topography and parts beyond it a series expansion of at least second order has to be considered:

$$\begin{aligned}\gamma_0(\varphi, h) = \gamma_0(\varphi) - (3.0877 \cdot 10^{-6} - 4.26 \cdot 10^{-9} \cdot \sin^2(\varphi)) \cdot h \\ + 7.2 \cdot 10^{-13} \cdot h^2 \quad \mathrm{m/s^2}\end{aligned} \qquad (5)$$

with h : height above reference surface

If eq.(5) is regarded under the aspect of usability in metrology it sticks out primarily by its simplicity and its adaptability all over the world. As a practical application Schwartz and Lindau [3],[4] describe "The New Gravity Zone Concept in Europe for Weighing Instruments under Legal Control" with a relative uncertainty level of 10^{-4}.

2.2 Point and mean gravity anomalies

Another representation of the gravity field comes with the introduction of gravity anomalies as the difference between gravity at an arbitrary level and the corresponding normal gravity. As a simple form the *point free-air anomalies* correlate the normal gravity with the gravity at the topographical surface.

$$\Delta g_F^N = g - \gamma_0(\varphi, h) \qquad (6)$$

Free-air anomalies depend strongly on height and are not suited for interpretation. The long and medium-wave part provided by global models, or corresponding mean anomalies, on the other hand, can be exploited, due to the smoothing of the high frequencies. Worldwide they vary about $\pm 4 \cdot 10^{-4}$ $\mathrm{m \cdot s^{-2}}$ with maximum values up to some 10^{-3} $\mathrm{m \cdot s^{-2}}$. Fig.2 gives an overview of the free-air anomalies in Western Europe derived from a 6'x10' grid of mean anomalies[5].

Bouguer anomalies are employed for regional and local investigations, as they are free from the effect of topography. The difference to eq.(6) is given by the plate reduction δg_P which introduces the density distribution in the Earth's crust and upper mantle.

$$\Delta g_B^N = \Delta g_F^N - \delta g_P \qquad (7)$$

Bouguer anomalies play an important role in geophysical prospecting.

Mean gravity anomalies over surface compartments are introduced in gravity field modeling, where the surface

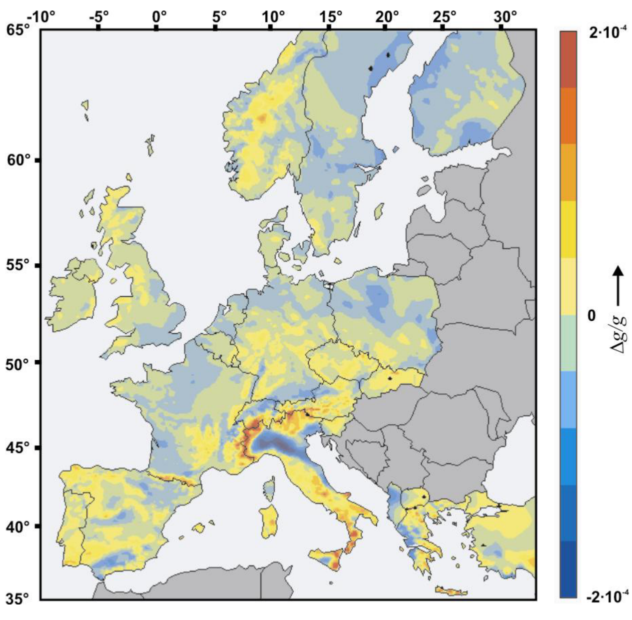

Fig. 2. Free air gravity anomalies in Western Europe based on a 6'x10' grid (≈11x20 km) of mean anomalies.

blocks generally are bounded by meridians and parallels. The mean values are calculated according to

$$\Delta\bar{g} = \frac{1}{\Delta\rho}\iint_{\sigma} \Delta g \, d\sigma \tag{8}$$

The block size $\Delta\sigma$ depends on the data distribution. The maximum gravity field resolution which can be achieved is $\sqrt{\Delta\sigma}$. This corresponds to a maximum degree of the spherical harmonic expansion $l_{\max}$ = 180°/resolution°. Gravity anomalies have been derived from several campaigns over land and sea and are to some extent freely available from international databases.

2.3 Spherical harmonic expansions

A common and proved method to represent the gravity field is the expansion of the gravitational potential into spherical harmonics [6]. The mathematical formulation bases upon the reciprocal distance between the *attracted* and the *attracting* point. The quality of this approximation is represented by the number of coefficients that have to be determined from equally as possible distributed gravity observations. The spatial resolution of the model is mirrored by the associated Legendre functions in degree and order. They produce a regular grid of surface elements on the unit sphere bounded by parallels and meridians, analogue to the geographical grid of the Earth (Fig.3). Considering the gravity field as a signal source the lower degrees represent the long-wave parts resulting from deeper structures.

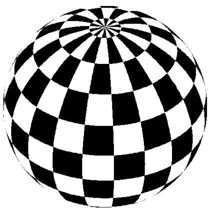

Fig. 3. Spherical harmonics on the unit sphere, example with degree and order (20,20)

As mentioned above the analytical expression of the potential has to meet eq.(3), that implies all masses inside the boundary. A reasonable predefinition for this bounding surface can be set by an approximation of the sea level. Therefor all topographic masses have to be reduced and integrated in the resulting model. This equipotential surface, denoted as *W,* is of special interest for the implementation of natural height systems.

Corresponding to the definition of gravity anomalies the potential *W* is generally related to the reference potential U_0. The so-called disturbing potential *T*, eq.(9), can be regarded as the residual field in terms of a statistical analysis :

$$T = W - U_0 \tag{9}$$

Expanded in spherical harmonics the disturbing potential can be computed from geopotential model coefficients as a function of the longitude λ and the polar distance ϑ , which is deduced from the geographical latitude.

$$T_l = \frac{GM}{a}\sum_{m=0}^{l}(\Delta C_{lm}\cos m\lambda + \Delta S_{lm}\sin m\lambda)P_{lm}(\cos\vartheta) \tag{10}$$

with

l,m : degree and order

ΔC, ΔS : spherical harmonic coefficients

$P_{lm}(\cos\vartheta)$: Legendre Polynoms with the argument of the polar distance ϑ

λ : longitude

We can apply eq.(2) in the same way to (9) and receive the gravity anomalies at *W*

$$\Delta g = \text{grad}\, T\,. \tag{11}$$

The partial derivation of *T* reads after some little modifications

$$\Delta g(r,\vartheta,\lambda) = \frac{1}{r}\sum_{l=2}^{\infty}(l-1)\left(\frac{a}{r}\right)^{l+1} T_l(\vartheta,\lambda). \tag{12}$$

The radial distance *r* allows a computation outside the boundary surface. Eq.(12) coincides with the definition of the free-air anomalies related to the level surface *W* .

2.4 Local gravity models

While spherical harmonic expansions only admit a global approach the limitation of areas for a local representation with high accuracy is also possible. Assuming a high-resolution grid of gravity data is available for the computation area, the disturbing potential *T* results from the integration of gravity anomalies over surface elements.

$$T(\vartheta,\lambda) = \frac{R}{4\pi}\iint_{\sigma} S(\psi)\Delta g \, d\sigma\,, \tag{13}$$

with R: radius of the approximated sphere.

The variable distance of the observations to the computation point $P(\vartheta,\lambda)$ is taken into account with the weighting function $S(\psi)$.

Contrariwise, with known potential respective geoid height *N* (*see* fig.5) the inversion of the integral formula, eq.(13), leads again to the gravity anomaly:

$$\Delta g_P = -\frac{\gamma_m}{R}N_P - \frac{\gamma_m}{16\pi R}\iint_{\sigma}\frac{N - N_P}{\sin^3(\psi/2)}d\sigma \tag{14}$$

An alternative approach to the integral formulas is enabled by statistical procedures, like the least square collocation.

2.5 Upward continuation

The procedures in 2.3 and 2.4 generally relate to the geoid as a common reference surface since an alternative reasonable representation of the gravity field is hardly conceivable. In the following step, each gravity value at the zero level has to be assigned to its equivalent at the physical surface. According to the schematic diagram in fig.4 the arbitrary surface point P differs from its foot at the geoid by the height *H*. Due to the definition of the geoid, *H* coincides with the height above sea level. Point heights can be derived from digital terrain models which are available at different grid widths.

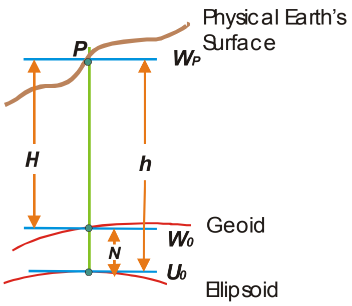

Fig. 4. Geometrical relations between the Earth's surface and virtual reference surfaces of the gravity field

Beside the height, especially the correct gravity gradient alongside the plumb line is required for the transfer of the gravity to the surface. After eq. (5) a linear respective square approach can be chosen for the normal gravity field. But the normal gradient may differ from realty by >10%, which leads to big errors in mountainous regions. Thus, the normal part has to be supplemented by an anomalous part:

$$\frac{\partial g}{\partial H} = \frac{\partial \gamma}{\partial H} + \frac{\partial(\Delta g)}{\partial H} \tag{15}$$

The solution of the anomalous part can be attributed to the first (Dirichlet) boundary-value problem of potential theory [5]. This may be solved by an integration over the surface gravity anomalies:

$$\frac{\partial(\Delta g)}{\partial H} = \frac{R^2}{2\pi} \iint_\sigma \frac{\Delta g - \Delta g_P}{l^3} d\sigma \tag{16}$$

After some transformations into spherical approximation the surface anomaly in P reads

$$\Delta g_P = \frac{R^2\left(r_P^2 - R^2\right)}{4\pi r_P} \iint_\sigma \frac{\Delta g}{l^3}\, d\sigma \tag{17}$$

where

R : radius of the approximating sphere

r_P : magnitude of the point vector P.

An efficient method of calculation is enabled by the 2D-Fast Fourier Transformation (FFT) [8]. This presumes a regular grid of anomalies on the boundary surface that is easy to produce from the gravity models. At first the grid at H=0 has to be converted into an image function $\Delta \tilde{g}(k_x, k_y, 0)$ via FFT, before the anomalies at H may be calculated in the frequency domain

$$\Delta \tilde{g}(k_x, k_y, H) = \Delta \tilde{g}(k_x, k_y, 0) \cdot e^{-|H|\sqrt{k_x^2 + k_y^2}} \tag{18}$$

With a subsequently inverse FFT the anomaly grid at the surface can be re-transformed into the space domain. But the FFT method should be applied more in extended areas so that long and short-wave parts are represented comparably. Also the maximum height has to be many times smaller than the extension of the area in order to not exceed the Nyquist frequency.

3. LIMITATIONS OF GENERIC MODELS

A mutual property of the described generic gravity field models is the basis of gravity observations. The occurrence, the distribution and the uncertainty of the origin data characterizes the quality of these models significantly. Comparatively dense gravity networks and databases only exist in the industry nations while large regions in the world show big data gaps. The availability of the origin data is indeed often restricted by the owner.

For the data acquisition over large areas currently only satellite missions and airborne gravimetric systems are qualified. Due to their limited resolution and precision both techniques contribute just long- and middle-wave parts of the gravity field. The high-frequent structures of the topography are supplied by terrestrial procedures which possess the required precision. Admittedly, they can be applied efficiently just in small regions. These facts, of cause, affect the quality and usability of today's databases in terms of metrologic applications.

Present gravity models employ area different types of datasets according to their procedure, their purpose and the covered area. Thus, the spatial resolution may vary between some 10° and about 1' (1 arc minute). The current international recommended geopotential model, the *EGM96*, comes out with a resolution of 0.5°×0.5° (≈ 54 km, degree and order: l,m=360,360). Ultra-high resolution models, e.g. the *GPM98*, base upon grids of up to 6'×6' anomalies (l,m=1800,1800), which coincide with a metric resolution of 10 km. The regional *EGG97*, limited to Europe, integrates datasets of 1'×1,5', identical with about 2×3.5 km.

Another limitation aspect of the described models arises from their geodetic nature. The main objectives can be found in the derivation of geoid heights or in the inverse interpretation of the Earth's structure. Though metrology/ physics may also profit from these results, actually they do not cover its basic needs.

The inverse computation of gravity anomalies from model parameters is always possible, but this is associated with a loss in accuracy comparing to the original data. This stems from filtering processes by smoothing and interpolation, to compensate irregularities, uneven distributions of observations or data gaps. Irregular distributed data between the grid knots suffer from the low resolution of the models.

The fig.4 demonstrates the effect of the limited resolution of two geopotential models for a small test area in the north of Germany. In fig.4 (a) an overview of the topography is given, including a low mountain range with heights up to 1000m.

The illustrations below show interpolated representations of free-air anomalies at the Geoid (coinciding with the sea level), which are results of the procedure described in 2.3. In this case free-air anomalies offer some important advantages: First, they can be computed directly, without the introduction of density hypotheses. Second, due to the condensation of all masses to the sea level they display, in a smoothed way, the structure of the above topography. For that purpose the contour lines with 200m equidistance have been overlaid.

The second graphic, fig.4 (b), represents interpolated free-air anomalies derived from the geopotential model *EGM96*. The resolution of this model only corresponds to 0.5°, which is the half of the test area extension. Considering the intensity of the interpolated field, the color gradient does not coincide with the contour lines. The maximum value is to be found somewhere at the foothills aside from the highest peak.

A significantly better accordance with the topography can be read out from the geopotential model *GPM98,* which is of 5 times higher resolution , fig.4 (c). In comparison to fig.4 (b) it has to accented that at the same number of colors the scale has been changed.

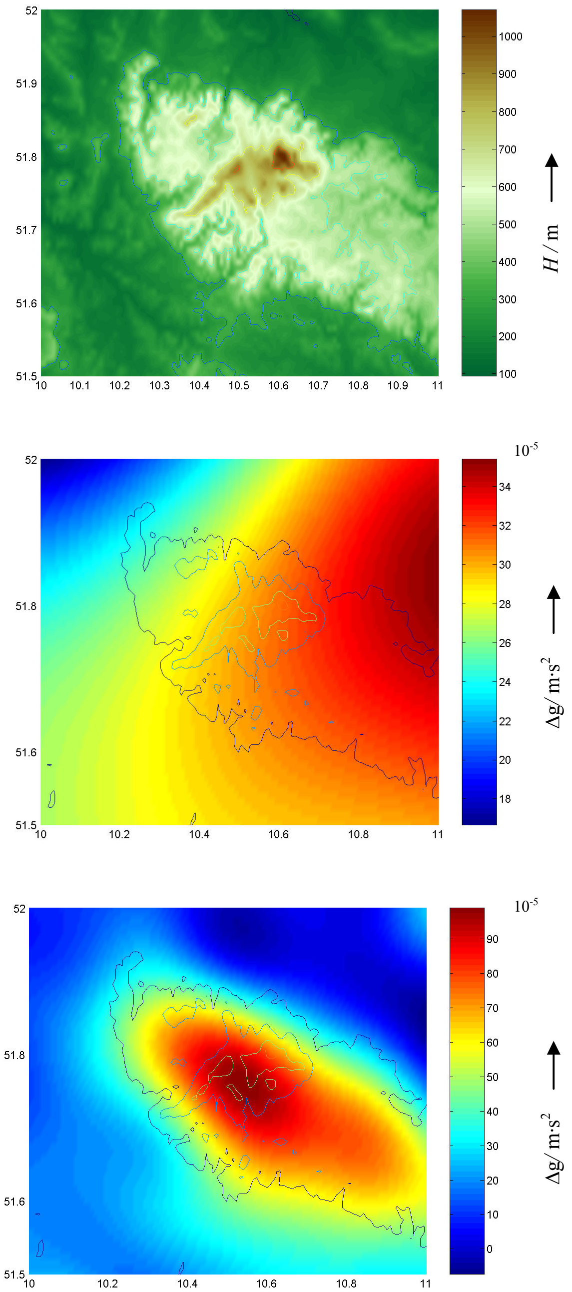

Fig. 5 (a) Digital Height Model of the test area in Northern Germany (b) free-air anomalies computed from the EGM96 geopotential model with overlaid contour-lines (c) free-air anomalies derived from GPM98 geopotential model

The maximum anomalies almost concentrate in the tops of the mountain range whereas the rest of the contour does not fit very properly. Beneath the move of the location the increase of the resolution has led to a triplication of the maximum value.

A quantified evaluation of the derived anomalies can be applied by observed reference gravity, which is hardly possible at the geoid. Before, the modeled anomalies have to be "upward continued" to the surface, as described in 2.5.

A study from Tscherning and Rapp [9] has demonstrated that the truncation of spherical harmonics at degree l_{max} produces an omission error due to the neglected part of the gravity field.

Accordingly for the *EGM96* this error reaches $\pm 2.5 \cdot 10^{-4}$ $m \cdot s^{-2}$ and for the *GPM98* , it is about $\pm 1.2 \cdot 10^{-4}$ $m \cdot s^{-2}$.

In order to decrease this error to dimensions $< 10^{-5}$ $m \cdot s^{-2}$, the models would have to be expanded to degree >10000.

Local gravity field models attain sometimes much better resolutions and lower error budgets. Usually the corresponding region is chosen on the basis of the available data to avoid undesirable data gaps or irregular distributions. But generally these models are not freely available, so that they are not part of this study.

The limited resolution of the available models may not be a handicap if they serve as the source for further developments, which have more synthetic character.

4. DIFFERENT STRATEGIES FOR SYNTHETIC GRAVITY MODELING

4.1 Synthetic fields based on spherical harmonics

The number of coefficients can be extended also artificially by scaling and recycling the coefficients of existing models. Therewith high-degree solutions, with a spatial resolution of up to 3.6 km [10], have been calculated especially to prove the accuracy of geoid models.

This is possible because of the self-consistency of the derived gravity parameters. A disadvantage of this method can be found in areas of an originally bad data distribution, where the solution is based on an unfavorable statistical estimation. With a limitation to well-known regions like Europe these errors can be kept small.

4.2 Synthetic fields based on mass-density-source models

Another strategy to determine the gravitational potential synthetically is to introduce information about external influencing factors. The integration of the differential mass elements can be solved by the knowledge of the mass-density-distribution within the interior of the earth and its topography. Through the numerical or analytical integration of Newton's integrals, self-consistent values of the gravitational potential and acceleration can be generated. Allasia [11], e.g., has developed analytic solutions for a continuous mass-density distribution.

A practical approach of a synthetic spherical harmonic model is given by Haagmans [12]. On the basis of the EGM96 he has expanded the number of coefficients by addition of a synthetic topography model. Therefor the potential of a digital terrain model with a spatial resolution of 9 km has to be transformed into spherical harmonics. The l_{max} of the synthetic model is determined by the resolution of the terrain grid. Due to the long-wave part of the topographic model is not very reliable it is replaced by the equivalent of the EGM96. The transition of both parts had to be low-pass filtered carefully. At present that model is mainly designed for geodetic purposes, like geoid computations.

There are still intensive ambitions to develop a detailed 3D-density model of the upper crust. The exact knowledge of the lithosphere's composition would be a first step on the way of a direct computation of gravity after Newton's law. But by now the generic gravity modeling improved fundamentally.

4. CONCLUSIONS

Modern physical applications influenced by the gravity field move in a wide range of uncertainty up to $<10^{-6}$. The required local gravity parameters can be derived from different representations of the gravity field Applications of low accuracy are covered by the compact model of the normal gravity field. Higher requirements have to be accomplished presently with local gravimetric measurements since gravity field models are of too low resolution. Auspicious approaches to the extension of the resolution can be found in the synthetic modeling. By this means the expensive local measurements may be replaced in the future by numerical methods for applications of relative accuracy requirements $> 5{\cdot}10^{-6}$.

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