XVII IMEKO World Congress
Metrology in the 3rd Millennium
June 22–27, 2003, Dubrovnik, Croatia

SIGNIFICANCE OF ORTHOGONALITY IN MASS CALIBRATION

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Abstract − The designs for submultiples of the unit of mass have been studied to find designs where two comparators are used. This study has found several cases where orthogonal designs can be used for using two comparators in a decade. Even though the orthogonal designs are for the case where the ratio of the comparator's standard deviation is 2, simulation showed extended values of the ratio could be used to reduce variances with the orthogonal designs.

Keywords: calibration design, orthogonality

1. INTRODUCTION

The efficiency of the methods of least squares depends substantially upon the choice of the calibration design[1,2]. Even though two comparators are used in a decade, much application has been paid to the design cases where only one comparator is used for comparison of mass standards of a decade series. Researchers have mentioned that when an orthogonal design for one comparator is used for the case of two comparators in a decade the orthogonal property is lost to produce covariances[3]. Therefore it is necessary to make sure that there are orthogonal designs for the case of two comparators. Also it is natural to extend the number of weighing to more than the usual 12 or 14 weighings in 10-5-2-2'-1 series as fully automated commercial mass comparators are available. The objective in the search for better designs is to those weighing designs that give a minimum value of variance or covariance with an increased number of weighing. Addition and repetition of elementary measurements increase the number of weighing.

General treatments of the least-squares method relevant to the present study have already been discussed [1,2,3,4,5]. Here the statistical procedure and notation of Cameron et al.[5] are applied. The decade series 10-5-2-2'-1 was studied to find designs that reduce the variances at increased number of weighing[6]. Here we consider that the case where two comparators of significantly different precision are used. In the least-squares method when two comparators are used, the variances due to the comparator of lower precision are further minimized than those due to the comparator of higher precision. To straighten this property, the concept of weighting is employed through a matrix, W, which has only diagonal elements which are $1, 1/\gamma_2^2, \ldots, 1/\gamma_n^2$ for n observations, where γ_n could be interpreted as a ratio of standard deviation to the comparator of higher precision. For example, the diagonal elements of $W^{-1/2}$ would be 1,1,1,1,1,1,1,1,2,2,2,2. Therefore design matrix X and observation matrix Y are transformed to weighted matrices, $Y^* = W^{-1/2}Y$ and $X^* = W^{-1/2}X$.

It was shown that the Matrix C introduced by Cameron et al.[5] which is conceptually close to $(X^{*\prime}X^*)^{-1}$ plays an important role because the variances and covariances of the estimates $\hat{\beta}$ can be expressed as follows:

$$\text{variance } (\hat{\beta}_i) = C_{ii}\sigma^2$$

$$\text{covariance } (\hat{\beta}_i \hat{\beta}_j) = C_{ij}\sigma^2$$

Here the variances and covariances of the estimates are directly proportional to C and to the variance of the comparator of n=1, σ^2. We use the term 'orthogonal' designs in this article to mean designs producing covariance of zero or C_{ij}=0. Therefore the criterion used in finding preferred designs was to minimize C using the method of least squares. Such designs could be evaluated by computation as in the previous study, where all the possible cases of weighing are considered [6].

2. CALIBRATION DESIGNS FOR WEIGHING

The elementary measurements in the series of 10-5-2-2'-1 are given in Table 1[6,7]. For simplicity the designs can each be expressed as a vector G whose elements are the repetition numbers of weighing in the corresponding row of Table 1.

Two comparators could be used in one of the following patterns according to their capacities.

Pattern A : (10, 5, 3, 2), (1)
Pattern B : (10, 5, 3), (2, 1)
Pattern C : (10, 5), (3, 2, 1)
Pattern D : (10), (5, 3, 2, 1)

Each pattern consists of 2 blocks where the second blocks employ comparators of better precision. As the pattern goes from A to D, the precision of the overall measurements are improved because more measurements are carried out by comparators of better precision.

The relations between "Elementary Measurements", Capacity, γ_n and Patterns for 10-5-2-2'-1 series are illustrated in Table 2 with γ_n of 2. The designs, where two comparators of different variances are used, are searched to give minimized sums of the absolute values of the elements in the inversion matrices of the normal equations, i.e. minimize $\sum\left|C_{ij}\right|$.

3. SEARCH RESULTS

The results of search have produced orthogonal designs that reduce the variances and covariances for the case where two comparators are used in a decade for submultiples of the unit of mass. The tested maximum number of weighing was 30 for all cases to find the designs. At first, tested weighting factor, γ_n, were 2, 3, 4, 5, 6, 7 and 8. Table 3 shows the designs which have orthogonality for 10-5-2-2'-1 series. Other values of γ_n rather than 2 did not produce orthogonality. In the tables "Sum C_{ij}" means the normalized sum of the absolute values of all the elements in the inverse matrices of the normal equations. The "Sum C_{ij}" could be interpreted as indicator of variance. And the normalization was done by the "Sum C_{ij}" of the orthogonal design of 12 weighings for 10-5-2-2'-1 series with γ_n of 1, G'{1,1,1, 1,2,2, 0,2,2 0}.

Pattern A's in Table 3 do not show any improvement because "Sum C_{ij}" remains to be 1 even with 2 comparators in a decade. As the pattern moves from A to D "Sum C_{ij}" reduces to 0.25. As expected, doubling the number of weighing of a design G' causes the "Sum C_{ij}" to become half of the original one. For example, the "Sum C_{ij}" of G'{4,4,4, 4,0,0, 4,2,2, 4} becomes 0.25.

Figure 1 shows the case where two comparators of Pattern C of 10-5-2-2'-1 series are tried with the design of G'={1,1,1, 1,2,2, 0,2,2, 0} which is for one comparator in a decade. The lower line indicates the sum of absolute values of off-diagonal elements of C_{ij}, and the upper line indicates the sum of absolute values of all the elements of C_{ij}. As the weighting factor γ_n increases on from 1 up to 8, orthogonality disappears as was indicated by Prowse [3] and the upper line shrinks by a little amount.

Figure 2 shows the case where two comparators of Pattern C of 10-5-2-2'-1 series are tried with the design of G'={2,2,2, 2,1,1, 0,1,1, 0} listed in Table 3-1. As the weighting factor γ_n increases from 1 and reaches to 2, the upper line, the sum of absolute values of all the elements of C_{ij}, shrinks to 0.5 and the lower line, the sum of absolute values of the off-diagonal elements of C_{ij}, reduces to zero. As the weighting factor γ_n further increases on up to 8, the upper line and lower lines become 0.47 and 0.09 respectively. Comparing Figure 1 and 2 we could find that the designs in Table 3-1 reduce the nonorthogonality by about 50 % and the over-all variances indicated by the upper lines by about 45 % when two comparators of significantly different precision are used in a decade.

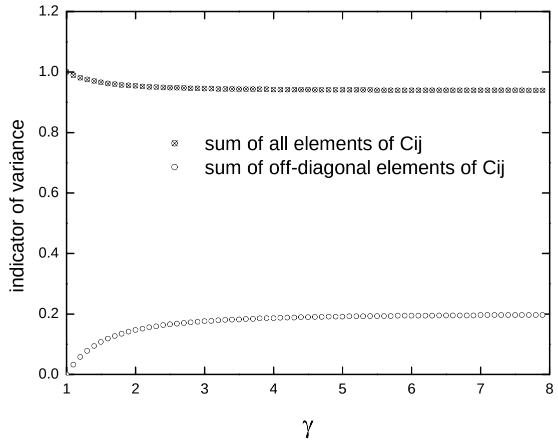

Figure 1. Application of the design for one comparator to the cases for two comparators

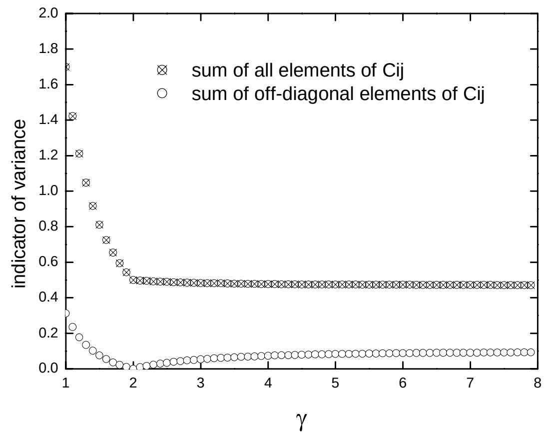

Figure 2. Application of the design for two comparator to the cases for two comparators

Table 1. Elementary measurements in each series

Series 10-5-2-2'-1						
Row	10	5	2	2'	1	1
1	1	-1	-1	-1	-1	
2	1	-1	-1	-1		-1
3		1	-1	-1	-1	
4		1	-1	-1		-1
5			1	-1	1	-1
6			1	-1	-1	1
7			1	-1		
8			1		-1	-1
9				1	-1	-1
10					1	-1

Orthogonal designs are limited only for the weighting factor γ_n of 2. However comparators could not be limited for the weighting factor γ_n of 2. Therefore an example design of nonorthogonality was tried to see the effect. For 10-5-2-2'-1 series and Pattern C of γ_n=4, a design of G' {2,2,2, 2,0,0, 1,2,0, 1} shows a minimum value of "Sum C_{ij}". The design of G' {2,2,2, 2,0,0, 1,2,0, 1} does not show any significant improvement compared to the orthogonal design with the weighting factor γ_n of 2 shown in Figure 2. To study the applicability of the orthogonal designs, simulation has been performed.

4. SIMULATION

The performance of the weighing designs were simulated and illustrated with standard deviations available from commercial comparators. We employed two comparators, one of maximum capacity 220 g, readability 10 μg and the other of maximum capacity 22 g, readability 1 μg, which have the standard deviations listed in Table 4. The standard deviations were obtained with 10 weighings at each capacity. The purpose of the simulation is to show the improvement in variance reduction with the orthogonal calibration designs when two comparators are used in a decade of 100-50-20-20'-10-10' g and to see the applicability of the designs over expanded range of the weighting factor γ_n.

The simulation was prepared with artificially generated elements of the observation matrix Y of all zeroes. Therefore it simplified in such a way that all the solutions, $\hat{\beta}_i$, had their exact nominal values respectively. The artificially generated elements of Y were white-noised in such a way that standard deviations in Table 4 were approximated with the standard deviations "within the group". The standard deviations "within the group" were calculated from the residuals between the observation with white noise and the estimated observation . The estimated observation could be determined by multiplication of design matrix and solution vector of the normal equation, $X\hat{\beta}$. The residuals were used in computing the variance employing the group variance of the residuals which convoluted the variance of both comparators. This approach is supposed to give a good estimate of the within-group standard uncertainty for the design and the two comparators.

Table 2. Relations between "Elementary Measurements", Capacity, γ_n of 2 and Pattern for 10-5-2-2'1 series

Row	10	5	2	2'	1	1	Capacity	γ_n			
1	1	-1	-1	-1	-1		10	1	1	1	1
2	1	-1	-1	-1		-1	10	1	1	1	1
3		1	-1	-1	-1		5	1	1	1	2
4		1	-1	-1		-1	5	1	1	1	2
5			1	-1	1	-1	3	1	1	2	2
6			1	-1	-1	1	3	1	1	2	2
7			1	-1			2	1	2	2	2
8			1		-1	-1	2	1	2	2	2
9				1	-1	-1	2	1	2	2	2
10					1	-1	1	2	2	2	2
							Pattern	A	B	C	D

Table 3. Orthogonal designs for 10-5-2-2'-1 series with 2 comparators

Pattern	Number of weighing	Sum C_{ij}	G'
A	12	1	1,1,1, 1,2,2, 0,2,2, 0
B	14	0.5	2,2,2, 2,0,0, 2,1,1, 2
	16	0.5	2,2,2, 2,2,2, 1,1,1, 1
	18	0.5	2,2,2, 2,4,4, 0,1,1, 0
C	12	0.5	2,2,2, 2,1,1, 0,1,1, 0
	14	0.5	2,2,2, 2,0,0, 2,1,1, 2
D	18	0.25	4,4,1, 1,2,2, 0,2,2, 0
	20	0.25	4,4,1, 1,1,1, 2,2,2, 2

Table 4. Standard deviations used for simulation

Comparator, max capacity & readability	Capacity, g	Standard deviation, μg
220 g, 10 μg	100	16
	50	10
	30	8
22 g, 1 μg	20	1.8
	10	1.6

The simulation is illustrated with Pattern B which approximates the distribution of standard deviations in Table 4. Simulations were performed 30 times. The discrepancies from the true values for the 50 g weight are plotted on Figure 4. Two cases are compared. The first case is with γ_n =1 and its corresponding design of G'{1,1,1, 1,2,2, 0,2,2, 0}. The second case is with γ_n =2 and its corresponding design of G'{2,2,2, 2,0,0, 2,1,1, 2}. Figure 4 shows the improvement when the orthogonal design of G'{2,2,2, 2,0,0, 2,1,1, 2} is used.

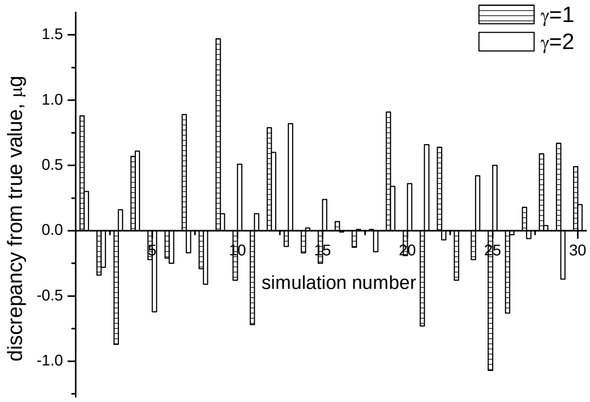

Figure 4. Discrepancies from true value in 30 simulations

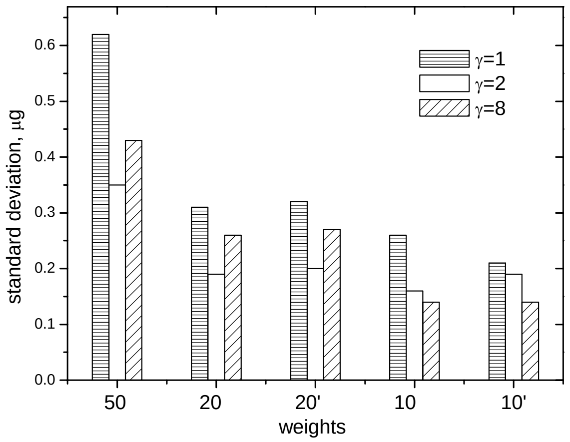

Figure 5. Standard deviations of 50, 20, 20', 10, 10' weights for designs with different γ_n

The standard deviations of the discrepancies for weights of 50, 20, 20', 10 and 10' g are plotted on Figure 5. In this figure the third case with γ_n =10 is included, which approximates the distribution of the standard deviation of Table 4. For Pattern B of 10-5-2-2'-1 series the best design with γ_n =10 is the same with that with γ_n =8. Therefore the third case is treated with γ_n =8 and its corresponding design of G'{3,2,2, 2,0,0, 0,1,1, 1}. The second case with the orthogonal design of γ_n =2 shows the best performance. Even though the third case approximates Table 4, the performance with γ_n =8 is not better compared to the second case with the orthogonal design of γ_n =2.

5. CONCLUSIONS

Orthogonal designs have been found where two comparators are used in a decade series of 10-5-2-2'-1 for submultiples of the unit of mass.

The orthogonal designs could reduce variances and covariances.

Even though the orthogonal designs are for the case where the ratio of the comparator's standard deviation is 2, simulation showed extended values of the ratio could be applied to reduce variances. This characteristic is believed to be derived from the orthogonal design.

REFERENCES

1. Prowse D. B., Anderson A. R., *Metrologia*, **10**, 123(1974).
2. Grabe M., *Metrologia*, **14**, 143(1978).
3. Prowse D. B., *Calibration of Masses*, APMP CSC(82) MS-21, Sydney, Australia, 1982, 33-54.
4. Bich W., Cox M. G., Haris P. M., *Metrologia*, **30**, 495(1993/4).
5. Cameron J. M., Croarkin M. C., Reybold R. C., *Designs for the Calibration of Mass Standards*, NIST Tech Note, 952, 1977.
6. Lee W.G., *Metrologia*, **34**, 365(1997).
7. Chapman G.D., *Orhtogonal Designs for the Calibration of Kilogram Submultiples*, National Research Council Report, No. AMS-002, 1991.

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