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A CERTAIN METHOD FOR CHEBYSHEV APPROXIMATION USED IN IOS METROLOGICAL SOFTWARE

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Abstract – The paper presents a method that is an attempt to solve the problem of non-linear approximation of geometrical elements according to Chebyshev norm. On the base of general two step algorithm it performs an evaluation of minimal deviation from maximal values of measured shapes driving to the optimal solution on the iterative way. The proposed method implemented for some geometric features has been verified for a many samples of simulated and measured datapoints.

Keywords: Chebyshev approximation, geometric features.

1. INTRODUCTION

Higher requirements faced to the coordinate measuring machines connected mainly with their coupling with the production processes and growing quality demands rise a necessity for producers of measuring software building new, accurate and efficient evaluation routines. The inspection of dimensional and tolerance specifications, expressed by technical documentation by means of CMMs leads to mathematical identification of geometric features associated with the actual, measured part and as a result to determination of its geometric parameters (the form, size and position deviations). The universal way of proceeding applied both to regular geometric features and freeform surfaces comprises of two steps. The first consists in fitting the ideal-form features for the sampled points, involving the approximation procedures, and followed by optimisation of the achieved results according to the established best fit criterion. Only then, in the second step the geometric deviations are calculated.

1.1. The problem definition

The approximation task may be formulated in a different way. In the regular geometric features measurements area it can be posed as bringing of the f_i function (2), represented by set of its values close to F function, given by known analytical formula (1):

$$F(\boldsymbol{x}, \boldsymbol{p}) = 0\,, \tag{1}$$

where: $\boldsymbol{x} = [x, y, z]^T$ is a vector of coordinates of datapoints; $\boldsymbol{p} = [p_1, ..., p_m]^T$ is a vector of parameters to be optimized; m =number of unknown parameters; the superscript T denotes the transposition.

$$f_i(\boldsymbol{x}, \boldsymbol{p}) = F(x_i, y_i, z_i, p_1, ..., p_m) \neq 0; i = 1, ..., n\,; \tag{2}$$

Thus the error of approximation is defined by (3).

$$e_u \overset{def}{=} \inf \left\| f_i - F \right\|. \tag{3}$$

F^* - element, satisfying the equation $e_u = \left\| f_i - F^* \right\|$ is called an optimal element.

It assumes that the measure of an approximation error is defined by a general L_p norm equation:

$$L_p = \left[\sum_{i=1}^{n} |r_i|^p \right]^{\frac{1}{p}}. \tag{4}$$

where: $r_i = \left\| f_i - F \right\|$ is the residual error between *i*th real datapoint and the ideal-form geometric feature fitted to these points, and p is an exponent: $p = 1, ..., \infty$, n =number of datapoint.

1.2. The Chebyshev norm

In coordinate measuring technique two kinds of norm function are most commonly used:

- in the event of $p = 2$ the L_2 norm is defined as a Euclidean length residuum.
- in the event of $p = \infty$ the L_∞ norm is defined, known as Chebyshev norm:

$$\left\| \mathbf{r} \right\|_\infty = \sup |r_i|\,. \tag{5}$$

A Chebyshev approximation (also commonly called "*minimax approximation*") minimizes the maximum distance between data and the approximating function:

$$\min_{\boldsymbol{p}} \max_i |r_i(\boldsymbol{p})| \Leftrightarrow \min_p \left\| f - F \right\|_\infty. \tag{6}$$

The use of Chebyshev norm is indicated in many cases where the residuals of the fit are known, in particular for all approximation of an empirical nature, where residuals are dominated by the inadequacy of the approximation rather than the errors of the measurement being approximated.

Furthermore, the use of Chebyshev norm is justified by the fundamental task of metrological software i.e. assessing the form deviations, according to minimum deviation zone

concept recommended by ISO 1011 [1] that is completely mathematically and geometrically interpreted.
In practice, the wide spread use of L_2 norm arises from facile and simple accomplishing of approximation methods results.
However, the minimax problems (6) in context of the non-linear, complicated functions of multiply variables bring to complex optimization problem, thus many algorithms have been developed for finding Chebyshev solution (6). To avoid the following hence inconvenience a computational method implementing technique of iterative improvement of optimal solution is suggested in this study.

1.3. Review of minimax algorithms

Over recent years several approaches have been considered and published [2]. Only some of them, which have had a significant impact on construction efficient methods along these ideas and have affected proposed method are quoted in the section below.
One of the earliest methods for solving the non-linear discrete minimax problem was the method provided by M. R. Osborne and G. A. Watson [3-4], proposed for the first time in 1969. In their method a linear approximation of the non-linear minimax problem was used, in accordance with (7).

$$\min_{\boldsymbol{t}} \max_{i} \left| f_i(\boldsymbol{p}^j) + f_i'(\boldsymbol{p}^j)^T \boldsymbol{t} \right| . \tag{7}$$

where: f_i^j denotes the gradient of f_i at the point $\boldsymbol{p}^j$ at the $j-th$ step iteration. The minimizer $\boldsymbol{t}^j$ was found implementing the linear programming. The theory given in the M. R. Osborne and G. A. Watson paper required that the successive matrix of f_i^j satisfied the Haar condition, and in that case convergence to a stationary point was established. In 1971 M. R. Osborne had relaxed the Haar condition assumption for the L_∞ algorithm and proved that the method was quadratically convergent. This method has been extended by D. H. Anderson and M. R. Osborne [5] in 1977, using the polyhedral norm formulation to provide an algorithm for general class of discrete non-linear approximaton problem, in particular L_1 and maximum norm problems. The length step of the minimizer $\boldsymbol{t}^j$ of (7) was adjusted during the iteration. The algorithm considered was similar to the Gauss-Newton method for the non-linear least squares problem.
Basing on the formulation of (7) K. Madsen [6] incorporated trust regions in the M. R. Osborne and G. A. Watson method. The linearized problem was solved there, subject to a local bound on the variable t. Above mentioned methods belong to first-order methods, what means that they are based on first derivatives only. They do provide fast convergence to the neighbourhood of a solution. In the event of the solution is singular the rate of convergence may become very slow.
There are many other methods with referring to second order approximation. One of them was elaborated by J. W. Bandler in 1985 [7] in the form of the non-linear minimax algorithm. It was a combination of the first order method and an approximate second order method. A quasi Newton method was used there to solve a set of non-linear equations that supplied the local solution. Switching criteria ensure the global convergence of this combined method.

2. THE MATHEMATICAL MODEL

The problem can be generally posed as assessment of $\boldsymbol{p}^*$ values satisfying the (6), where the value of residuum $r_i(\boldsymbol{p})$ is expressed by (8).

$$r_i(\boldsymbol{p}) = \left| f_i - F_i(\boldsymbol{p}) \right| ; \; i = 1,...,n . \tag{8}$$

The basis of the method is bringing the problem of non-linear approximation to a sequence of equivalent linear problems satisfying the conditions provided by M.R. Osborne and G. A. Watson model [3-4].

2.1. Assumptions

A. The function (1) is sufficiently smooth and has continuous partial derivatives including second order in the neighbourhood and at a given point $\boldsymbol{p}$.
If assumption is valid function the $F(\boldsymbol{p})$ can be then expanded into Taylor series in local neighbourhood $\boldsymbol{p}$ (9).

$$F_i(\boldsymbol{p} + \boldsymbol{\delta p}) = F_i(\boldsymbol{p}) + \nabla F_i(\boldsymbol{p})\boldsymbol{\delta p} + 0(\|\boldsymbol{\delta p}\|^2) , \tag{9}$$

where : $\nabla F_i = \dfrac{\partial F_i}{\partial p_k} =$ ith- row of the Jacobian matrix: $\boldsymbol{M} = \nabla F$; $k = 1,...,m$; $0(\|\boldsymbol{\delta p}\|^2)$ =omitted higher order elements.
B. The matrix $\boldsymbol{M}$ has full rank m.
Substituting expression (9) for $F(\boldsymbol{p})$ in equation (8) follows to the set of n linear equations having m unknown quantities $\boldsymbol{\delta p}$ thus the non-linear optimization problem reduces to the linear one.

2.2. An algorithm

The task of optimization is solved iteratively by an algorithm incorporated principally in two stages, which is a modification of the usual Newton method for finding a zero of a vector function of several variables. This method combines the rapid local convergence of classic Newton's method with a globally convergent strategy that ensures some progress towards each iteration.
Step 1 of iteration: Calculation of $\boldsymbol{t}$ vector to minimize (10).

$$\left\| f(p^j) - F(p^j) - \nabla F(\boldsymbol{p}^j)\boldsymbol{t} \right\| . \tag{10}$$

This basic step of Newton method, which formally applies to any norm determines a direction of progress $\boldsymbol{t}$. In the original M. R. Osborne, G. A. Watson, D. H. Anderson method the $\boldsymbol{t}$ is obtained by linear programming, however this way may be occasionally insufficient and can provide afterwards convergence to not a stationary point [7]. To avoid these drawbacks an alternative that deals with

minimization problem (10) for calculating such $t^j - th$ moving a residuum $|r - Mt|$ closer enough to minimum by applying a singular value decomposition approach known as SVD technique [8] (that explicitly constructs orthonormal bases for the null spaces and range of a matrix).
The SVD technique assumes that any nxm matrix M, whose number of rows is greater then or equal to number of columns $(n \geq m)$ can be expressed as a product of an orthogonal matrix $\mathbf{U}(nxm)$, a diagonal matrix W and the transpose of an orthogonal matrix $V(nxn)$ (11).

$$(M) = (U) * \begin{pmatrix} w_1 & & & \\ & w_2 & & \\ & & \dots & \\ & & & w_m \end{pmatrix} * (V^T). \qquad (11)$$

Rearranging (11) the method finds the solution t closest in the least square sense among all possible values, so that with nonsquare matrices looks like this:

$$(t) = (V) * \begin{pmatrix} 1/w_1 & & & \\ & 1/w_2 & & \\ & & \dots & \\ & & & 1/w_m \end{pmatrix} * (U^T) * (r). \qquad (12)$$

This variant is relatively easy to minimize and can be used no matter haw singular the matrix M is and whether it is "almost" unique.
At the point p^j the value of t^j solves the (10), and let a result be given by (13).

$$\hat{h}^j = \min_t \left\| f(p^j) - F(p^j) - \nabla F(p^j)t \right\|; \text{ with } t = \delta p \quad (13)$$

- Step 2 of iteration: Calculating of an optimal step coefficient γ from the range $(0<\gamma<1)$ to minimize (14).

$$\left\| f(p^j) - F(p^j + \gamma \delta p^j) \right\|. \qquad (14)$$

To determine a suitable value for γ^j let $\theta \in (0,1)$ be given and be defined a set: $\Pi = \{1, \theta, \theta^2, \dots\}$.
Let the function $\psi(p^j, \gamma)$ be defined, such that:

$$\psi(p^j, \gamma) = \frac{h^j - h^{j+1}}{\gamma(h^j - \hat{h}^j)}; \text{ where:}$$

$$h^j = \left\| f(p^j) - F(p^j) \right\|, \text{ and}$$

$$h^{j+1} = \left\| f(p^j) - F(p^j + \gamma \delta p) \right\|.$$

Then, it is possible to choose a step length γ^j as a largest element in Π satisfying (15).

$$\psi(p^j, \gamma) \geq \tau, \qquad (15)$$

where: τ is any fixed value satisfying: $(0 < \tau < 1)$, independent of j.

A point p^* is a stationary point of $\|f - F(p)\|$, if:

$$\left\| f - F(p^*) \right\| = h^* = \min_t \left\| f(p^*) - \nabla F(p^*)t \right\|. \qquad (16)$$

It results from the above definition, that:

$$\left\| f - F(p^j) \right\| \geq \hat{h}^j. \qquad (17)$$

If the algorithm does not terminate in a finite number of iterations, then the sequences of $\{h^j\}$ is convergent.
The proof quoted from [5] is as follows: If p^j is not a stationary point $\|f - F(p^j)\|$ it follows from (17) that:

$$\hat{h}^j < \left\| f - F(p^j) \right\|.$$

From (15) it leads to (18).

$$h^{j+1} \leq h^j - \tau\gamma^j(h^j - \hat{h}^j), \qquad (18)$$

so that the sequence $\{h^j\}$ is decreasing and bounded below, and hence converges.

- Step 3 of iteration: If γ^j has an appropriately chosen value the values of p parameters are updated respectively:

$$p_k^{j+1} = p_k^j + \gamma^j \delta p^j. \qquad (19)$$

During the iterations the estimated values of p_k^{j+1} are used for building a new Jacobian matrix for "step 1" of the described algorithm.
On the input the values of vector p^o specifies the initial guess for the first recursion of algorithm. The sequence 1^0 to 3^0 steps is continued until the convergence will have been achieved i.e. the normalized residual will become less than the specified tolerance or if algorithm will not reach the specified convergence in the maximum number of iteration.

4. THE IMPLEMENTATION AND RESULTS

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The method was implemented for some several geometric elements derived from a standard software package of IOS coordinate measuring machines. The essential mathematical routines were developed used Delphi Borland tools.
The implementation was applied under numerical tests involving generated reference data sets and then according to the empirical data sets from coordinate measurements.
Generating data have been treated similarly to the created reference for Gaussian approximation problem taking into cosideration sufficiency conditions for a solution to the minimax approximation problem (i.e. for Chebyshev fitting circle the datapoints lie between two concentric circles each other apart from an overall form deviation) [9].
As an example some results of circle fitting problem is revealed below. Before all others the procedure has been checked for simulated circular profiles. Calculations have

been performed for several 100-datapoints samples. Since time and number of recursions are the crucial factors affecting on effectiveness of the method they were particularly under examination. Then the calculation results have been compared to the results of standard Gaussian procedures. These latter tests show that the number of recursions are not affected considerably by disadvantageous choice of starting point although the Chebyshew iteration is more time consuming than least squares procedure.
Algorithm convergence and its numerical effectiveness depend mostly on determination of proper direction of optimisation $\boldsymbol{t}$ and the choice of appropriate coefficient step γ .

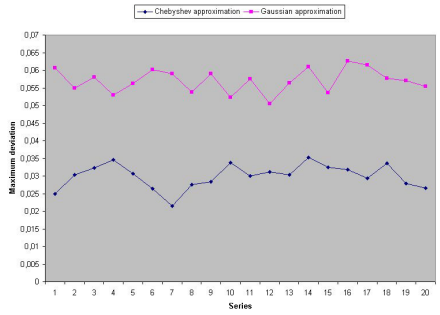

Fig. 1. The results of comparison of maximum deviation values assessed from Gaussian and Chebyshev approximation for sample m=100 datapoints.

It can be observed also that the assessed values of geometric maximum deviations are significant smaller than those obtained using Gaussian approximation and can differ up to 40% as “Fig. 1” presents (20 seriees of calculations).
Finally, the procedure has been applied for samples of experimental datapoints.

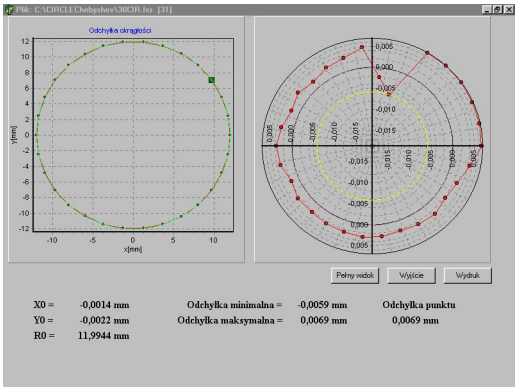

Fig. 2. The result screen of Chebyshev approximation for experimental sample m=30 data datapoints.

The “Fig. 2” and “Fig. 3” show the graphic result screens to Chebyshev and Gaussian approximation respectively, which expressively demonstrate how these two fitting techniques considerably distinguish one from the other when the data contains “wild” point. One argues that the L_2 norm does not seem to be a suitable measure of goodness of fit in these cases.

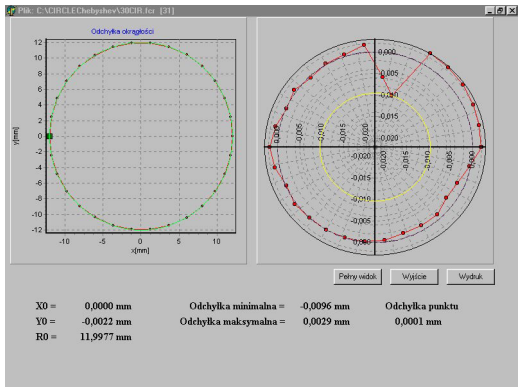

Fig. 3. The result screen of Gaussian approximation for experimental sample m= datapoints.

5. CONCLUSION

The method has been proposed is an attempt to solve a discrete minimax approximation problem. It has been established generally on M. R. Osborne, G. A. Whatson, D. H. Anderson strategy. Through certain simplifications (i.e. linearisation of approximating problem) it allows use of the linear algebraic numerical tools. It has been implemented to some geometric elements and tentative numerical tests have proven this software successful in solving of data fitting problems, however it is essential to continue its verification and to access its numerical stability and accuracy.

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