The Range of Applying Approximate Methods of Expanded Uncertainty Estimation in Indirect Measurements

Przemysław OTOMAŃSKI
Institute of Electronics and Telecomunications, Poznan University of Technology
Piotrowo 3a str., 60-965 Poznań, Poland

ABSTRACT

The results of examining error evaluation of the coverage factor with approximate methods in indirect measurements have been presented in the paper. Basing on the method's accuracy analysis, the scope of application of the approximate methods of the expanded uncertainty estimation has been presented; the criterion of not-exceeding a preset value of the error has been implemented. The knowledge of coverage factor characteristics for the convolution of three selected probability distributions was used for the examination.

Keywords: expanded uncertainty, coverage factor, probability distribution.

1. INTRODUCTION

The estimation of expanded uncertainty of a measurement result is always an approximate estimation. When deciding about the method of expanded uncertainty estimation, one should be aware of the effects of choosing a particular method from the viewpoint of its accuracy. The basis for estimating the accuracy of applied approximate methods of the estimation of expanded uncertainty is the assumption on the necessity the assessment methods, which could be regarded as exact estimation.

An essentially appropriate concept was adopted, which is taken into consideration, that the method based on the command of the convolution of probability distributions of errors of components may be regarded as an exact method. Due to complexity and time-consuming character of computing the convolution of many distributions of components, the results of such computing are, in general, hardly ever published. Therefore, approximate methods are generally accepted and recommended.

There are the results of publications [1], [2], [3], concerning the analysis of accuracy of approximate methods of expanded uncertainty estimation for simple direct measurement, when there are only two component standard uncertainties.

In the present paper the analysis of accuracy of estimating the coverage factor in indirect measurements was described. The examination results for the convolution of two Student distributions and one rectangular distribution were described.

In the case considered, in indirect measurements, the expanded uncertainty is determined as:

$$u = k(\alpha) \cdot u_{\bar{c}} \tag{1}$$

where, according to the uncertainty propagation law, the combined standard uncertainty $u_{\bar{c}}$ for a quantity measured indirectly when variables are independent:

$$u_{\bar{c}} = \sqrt{\sum_{j=1}^{N} \left(\frac{\partial f}{\partial x_j} \right)^2 u_{c_j}^2} \tag{2}$$

In this situation the coverage factor $k(\alpha)$ acquire values of standardized variable of distributions being convolution of distributions of which standard deviation are j-th values of standard uncertainties. As values of standardized variables of convolutions are hardly ever published, the unknown coverage factor is evaluated with approximated method. It causes specific errors of estimate of expanded uncertainty.

On the presumption, that knowledge of convolution of component distributions permits to estimate expanded uncertainty with strict accuracy it is assumed, that error, of which absolute values is described by relationship below will be measure of discrepancy between approximate and exact method:

$$\delta = \frac{| u - u_e |}{u_e} \cdot 100\% \tag{3}$$

where u - it is expanded uncertainty evaluated by means of approximate method

$$u = k(\alpha) \cdot u_c \tag{4}$$

u_e - it is expanded uncertainty estimated "exactly", on the basis of knowledge of distribution, which in the measuring event is the convolution of two Student distributions and one rectangular distribution. The coverage factor $k_{S_1 S_2 R}(\alpha)$ could be regarded as exact value:

$$u_e = k_{S_1 S_2 R}(\alpha) \cdot u_c \tag{5}$$

In the most of considered measuring events error described by the dependence (3), will be error the estimate of unknown the coverage factor value $k(\alpha)$, which assumes form:

$$\delta = \frac{\left| k(\alpha) - k_{S_1S_2R}(\alpha) \right|}{k_{S_1S_2R}(\alpha)} \cdot 100\% \qquad (6)$$

Using different approximate methods estimation of expanded uncertainty, different values should be assigned to the coverage factor $k(\alpha)$. The results of experiments for three approximate methods estimate of expanded uncertainty: method of imposed values, method of effective number degrees of freedom, method of geometrical sum have been presented in the paper.

2. CHARACTERISTICS OF THE CONVOLUTION OF TWO STUDENT'S DISTRIBUTIONS AND ONE RECTANGULAR DISTRIBUTION

A measuring event, which utilizes a convolution of two Student distributions and one rectangular distribution is an example of indirect measurement carried out by means of two measuring devices, which, in case of repeated measurements, show a scatter of results, a type-B standard uncertainty of one of the devices can be neglected, and the number of measurements is small (n<30). Therefore, three standard uncertainties are analyzed: type-B standard uncertainty, which reflects a standard deviation of rectangular distribution and two type-A standard uncertainties, which reflect a standard deviation of Student distribution.

On the basis of the developed analytical description of coverage factors in case of the analyzed convolutions one is able to identify all parameters, whose function are the factors. One is able to demonstrate that a coverage factor for the convolution, from now on referred to as factor $k_{S_1S_2R}(\alpha)$ is a function of 5 variables: probability α, number of degrees of freedom $m_1 = n_1 - 1$ and $m_2 = n_2 - 1$ first and second Student's distributions and the ratio of standard uncertainties η_S and η:

$$k_{S_1S_2R}(\alpha) = f(\alpha, m_1, m_2, \eta_S, \eta) \qquad (7)$$

where:

η_S - is the ratio of standard uncertainties of type A

η - is the ratio of combined standard uncertainty of type A to standard uncertainty of type B.

3. COMPUTATIONAL RESULTS OF THE COVERAGE FACTOR FOR THE CONVOLUTION

Calculations were executed for one probability value $\alpha = 0{,}99$, for small values m and for the value series η, ranging from 0,1 to 10.

Matlab program was used for the calculations and the following were assumed:

- approximation accuracy of the probability range α over the variable k, ε =1e-4,
- the number of integration ranges in the Simpson's method of integration 300
- multiple j =20

computational results are presented in table 1.

Table 1. Values of the coverage factor $k_{S_1S_2R}(\alpha)$

$1/\eta$	η	$m_1 = 3$	$m_1 = 9$
		$m_2 = 3$	$m_2 = 9$
10		1,8552	1,7969
5		2,1211	1,9313
4		2,2688	2,0016
3		2,5203	2,1125
2		3,0548	2,3227
1	1	4,2031	2,7195
	2	5,0938	2,9812
	3	5,3500	3,0475
	4	5,4500	3,0788
	5	5,4875	3,0900
	10	5,5750	3,1150

Characteristics of the coverage factor are presented in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse.

Characteristics of the coverage factor $k_{S_1S_2R}(\alpha)$ are compared to the characteristics of the coverage factor $k_{SR}(\alpha)$ for the convolution S*R and to the value of the coverage factor $k_N(0.99)$ for a normal distribution.

Fig. 1 shows the characteristics of the coverage factor $k_{S_1S_2R}(0.99)$ for $m_1 = m_2 = 3$, $m_1 = m_2 = 9$, $\eta_S = 1$ in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse.

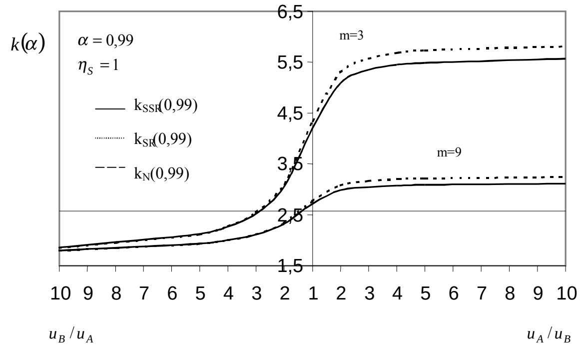

Fig. 1 Characteristics of the coverage factor $k_{S_1S_2R}(0{,}99)$ for $m_1 = m_2 = 3$, $m_1 = m_2 = 9$, $\eta_S = 1$ and $k_{SR}(0{,}99)$ in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse.

In this situation both samples have the same number of degrees of freedom. Broken line shows characteristics of the coverage factor $k_{SR}(0.99)$ for the convolution S*R, for $m = 3$ and $m = 9$, and the coverage factor $k_N(0.99)$ for a normal distribution.

The characteristics of factors $k_{S_1S_2R}(\alpha)$ clearly approach the characteristics of the factor $k_N(\alpha)$ for a normal distribution. Moreover, in accordance with the central limit theorem, the characteristics of the coverage factor $k_{S_1S_2R}(\alpha)$ and $k_{SR}(\alpha)$ clearly trend to approach the value of the factor $k_N(\alpha)$ as the sample size increases. The phenomenon is observed in the domain where $u_A > u_B$, further called domain A. Whereas in the domain where $u_B > u_A$, further called domain B, the influence of the sample size is much smaller and fades as the values of the ratio $u_B > u_A$ increases.

4. THE METHOD OF GEOMETRICAL SUM

A method called the method of geometrical sum is utilized quite often in practical measurements. According to this method the expanded uncertainty is estimated as a geometrical sum of the component expanded uncertainties:

$$u = \sqrt{\sum_{j=1}^{N} u_j^2} \tag{8}$$

For indirect measurement the expanded uncertainty is calculated according to the relation presented below, taking into account that in the analyzed case of the convolution of three component distributions S*S*R, the expanded uncertainty will be equal to:

$$\begin{aligned} u &= \sqrt{k_{S_1}^2(\alpha)\cdot u_{A_1}^2 + k_{S_2}^2(\alpha)\cdot u_{A_2}^2 + k_R^2(\alpha)\cdot u_B^2} = \\ &= k_{gSSR}(\alpha)\cdot u_{\bar{c}} \end{aligned} \tag{9}$$

Assuming that all partial derivatives in the expression $u_{\bar{c}}$ are equal to one, the coverage factor estimated by means of a method of geometrical sum from now on referred to as $k_{gSSR}(\alpha)$, will assume the form:

$$k_{gSSR} = \frac{\sqrt{k_{S_1}^2(\alpha)\cdot u_{A_1}^2 + k_{S_2}^2(\alpha)\cdot u_{A_2}^2 + k_R^2(\alpha)\cdot u_B^2}}{\sqrt{u_{A_1}^2 + u_{A_2}^2 + u_B^2}} \tag{10}$$

After appropriate substitutions, factor $k_{gSSR}(\alpha)$ expressed as a function of standard uncertainties ratios η and η_S and specific of coverage factors, will be produced:

$$k_{gSSR} = \sqrt{\frac{\eta^2\left(k_{S_1}^2(\alpha)\cdot \eta_S^2 + k_{S_2}^2(\alpha)\right) + \left(\eta_S^2+1\right)\cdot k_R^2(\alpha)}{\left(\eta^2+1\right)\left(\eta_S^2+1\right)}} \tag{11}$$

Fig. 2 presents characteristics of coverage factor $k_{gSSR}(0.99)$ and $k_{SSR}(0.99)$ in the function η and its converse.

A characteristic feature of the computed factor $k_{gSSR}(\alpha)$ is that in domain A its values will not differ significantly from the values of factor $k_{SSR}(0,99)$, which are exact values. With the increasing number of degrees of freedom m, the differences diminish. In domain B the values of the analyzed factor only to a lesser extent depend on the number of degrees of freedom and the values of factors $k_{SSR}(0,99)$ are getting close to the values of factor $k_R(0,99)$ for a rectangular distribution.

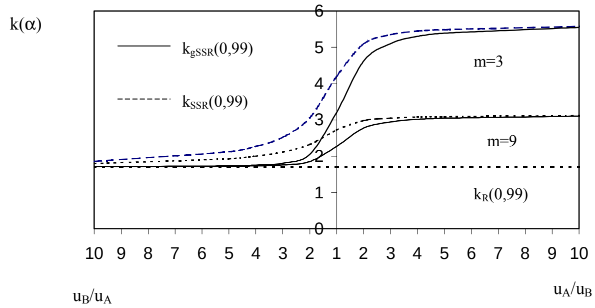

Fig. 2 Characteristics of the coverage factor $k_{gSSR}(0.99)$, $k_{SSR}(0.99)$ for $m_1 = m_2 = 3$, $m_1 = m_2 = 9$ in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse

According to the assumption that the value of coverage factor for the analyzed convolution of component distributions may be regarded as an exact value, the absolute value of error estimation by means of this approximate method is defined as:

$$\delta = \frac{\left| k_{gSSR}(0.99) - k_{S_1S_2R}(0.99)\right|}{k_{S_1S_2R}(0.99)}\cdot 100\% \tag{12}$$

Fig. 3 presents absolute error values δ of factor estimations $k_{gSSR}(0.99)$ against errors δ' of the factor estimation $k_{gSR}(0,99)$ in the function η for various values of m_1 and m_2, where:

$$\delta' = \frac{\left| k_{gSR}(0.99) - k_{SR}(0.99)\right|}{k_{SR}(0.99)}\cdot 100\% \tag{13}$$

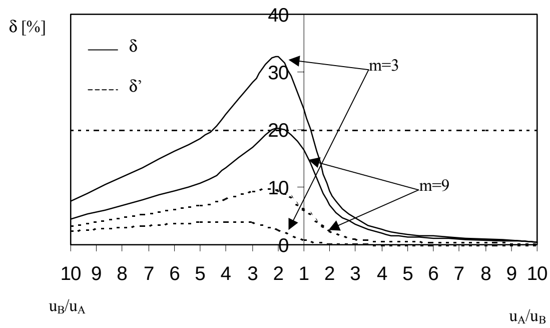

Fig. 3 Absolute error values δ of factor estimations $k_{gSSR}(0.99)$ and errors δ' of the factor estimation $k_{gSR}(0,99)$ in the function η and its converse

According to Fig. 3 there is a limitation of utilization of the method of geometrical sum especially in the domain where the ratio of standard uncertainties η is close to one.

In spite of a relatively big increase of errors in case of indirect measurements in comparison with errors in direct measurements, they exceed the assumed value of 20% in a small range of changes of standard uncertainties ratio for the number of degrees of freedom $m = 3$.

The present results of research, which concern the trend of changes of errors of coverage factor estimation are connected with characteristics features of coverage factors $k_{gSSR}(0.99)$ estimated by means of the method of geometrical sum and the characteristics of the coverage factor $k_{S_1S_2R}(0.99)$ for the analysed convolution. Analysis of the method of geometrical sum shows that this is a method of coverage factor estimation, whose error in a wide range of changes in standard uncertainties ratio does not exceed the assumed value of 20%.

5. THE METHOD OF EFFECTIVE NUMBER DEGREES OF FREEDOM

Document [4] suggests for a measuring event with a small number of tests a method according to which the coverage factor $k(\alpha)$ assumes the values of standardized variable of Student distribution $k_{S_e}(\alpha)$, read from the table of this distribution for the method of effective number degrees of freedom m_e.

According to Welch-Satterthwaite Formula [2], if the combined standard uncertainty is a root of a sum of two or more variances estimated on the basis of results of not numerous test with unknown standard deviation σ, the unknown distribution of the required standardized variable can be approximated by means of a Student distribution for the effective number of degrees of freedom m_e. In the considered case of indirect measurement the effective number of degrees of freedom is described by means of relationship (14) resulting from the general Welch-Satterthwaite formula:

$$m_e = \frac{u_{\bar{c}}^4}{\sum_{j=1}^{N} \frac{1}{m_{A_j}} \left(\frac{\partial f}{\partial \overline{x}_j} \right)^4 \cdot u_{A_j}^4 + \sum_{j=1}^{N} \frac{1}{m_{B_j}} \left(\frac{\partial f}{\partial \overline{x}_j} \right)^4 \cdot u_{B_j}^4} \qquad (14)$$

where:

- $u_{\bar{c}}$ is a standard combined uncertainty of value Y measured indirectly and computed according to the uncertainty propagation law
- $m_{A_j} = n_j - 1$ is the number of degrees of freedom of the j-th measurement
- m_{B_j} is the number of degrees of type B freedom and is computed on the basis of reliability of component standard uncertainty of type B.

In a situation when type-B standard uncertainty is estimated on the basis of known rectangular distribution whose borders are defined by the limiting error of measuring devices, one can assume that this uncertainty is well known. Therefore, for the following analysis one can assume that the relative uncertainty of type B values equal to 0.1 [1,4], which reflects the number of degrees of freedom m_B =50. Assuming that all partial derivatives are equal to one, and after all appropriate transformations for the analyzed situation one obtains:

$$m_e = \frac{\left(\eta^2 + 1\right)^2}{\frac{\eta^4}{\left(\eta_S^4 + 1\right)^2} \left(\frac{1}{m_1} \cdot \eta_S^4 + \frac{1}{m_2} \right) + \frac{1}{m_B}} \qquad (15)$$

This form of relationship describing m_e permits to present the characteristics of coverage factor of Student distribution for the effective number of degrees of freedom $k_{m_eSSR}(\alpha)$ in function η for various values m, in order to compare it with the characteristics of coverage factor for the analyzed convolution $k_{S_1S_2R}(\alpha)$. Fig. 4 presents the characteristics of coverage factor $k_{m_eSSR}(0.99)$, $k_{S_1S_2R}(0,99)$ for $m_1 = m_2 = 3$ as well as for $m_1 = m_2 = 9$, $\alpha = 0,99$ and η_S =1.

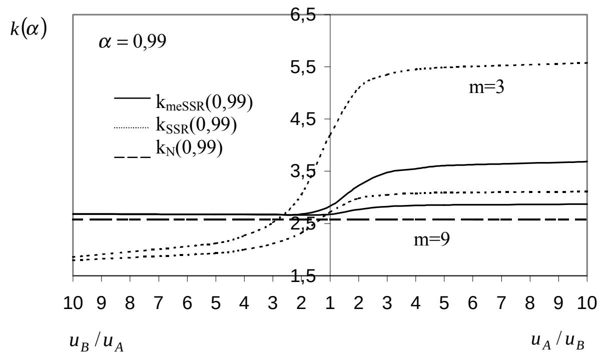

Fig. 4 Characteristics of the coverage factor $k_{m_eSSR}(0.99)$, $k_{SSR}(0.99)$ for $m_1 = m_2 = 3$, $m_1 = m_2 = 9$ in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse

The characteristic feature of the computed factor $k_{m_eSSR}(\alpha)$ is such that in domain A its values differ considerably from the value of factor $k_{S_1S_2R}(\alpha)$ which are exact values. With the increasing number of degrees of freedom m, the differences diminish. Observation of the same characteristics for convolution S*R shows good convergence of the characteristic features of $k_{m_eSR}(\alpha)$ and $k_{SR}(\alpha)$ in domain A. In domain B the factor assumes constant values independent from the number of degrees of freedom m and the values are close to the values of the factor $k_N(0,99)$ for a normal distribution [5].

According to the assumption that the value of coverage factor for the analyzed convolution of component distributions may be regarded as an exact value, the absolute value of error estimation by means of this approximate method is defined as:

$$\delta = \frac{\left| k_{m_e SSR}(0.99) - k_{S_1 S_2 R}(0.99) \right|}{k_{S_1 S_2 R}(0.99)} \cdot 100\% \qquad (16)$$

Fig. 5 presents absolute error values δ of factor estimations $k_{m_e SSR}(0.99)$ against errors δ' of the factor estimation $k_{m_e SR}(0{,}99)$ in the function η for various values of m_1 and m_2, where:

$$\delta' = \frac{\left| k_{m_e SR}(0.99) - k_{SR}(0.99) \right|}{k_{SR}(0.99)} \cdot 100\% \qquad (17)$$

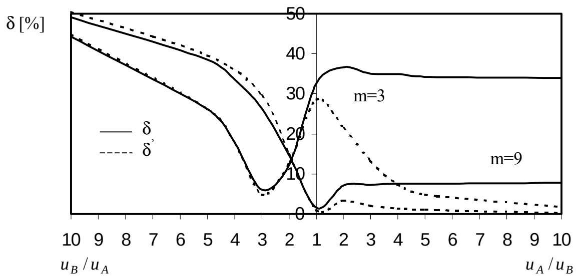

Fig. 5 Absolute error values δ of factor estimations $k_{m_e SSR}(0.99)$ and errors δ' of the factor estimation $k_{m_e SR}(0{,}99)$ in the function η and its converse

According to Fig. 5 there is a limitation of applying the method of effective number of degrees of freedom for not numerous population – m=3 in a situation when we are in a domain type A. With the increasing m, the value of error decreases considerably. The results of research, which are presented in the present paper indicate only the trend of changes of errors of coverage factor estimation, according to the recommended by the international document [4] approximated method of assessment. In spite of the fact that this is the method recommended by the international document for estimating the coverage factor in indirect measurements, when short series are available, this method has strong limitations from the viewpoint of its accuracy.

6. THE METHOD OF IMPOSED VALUES

The international document recommends among other things to use arbitrarily imposed values of coverage factors which are equal appropriately $k(\alpha)$=3 for α=0.99.

According to this assessment, factor $k(\alpha)$ is attributed the value of the factor, which are close to the values of standardized variable of normal distribution. In the analyzed situation of indirect measurements, the absolute values of errors resulting from estimation of coverage factors by means of the method of imposed values are described in the following manner:

$$\delta_{k_1} = \frac{\left| 3 - k_{S_1 S_2 R}(0.99) \right|}{k_{S_1 S_2 R}(0.99)} \cdot 100\% \qquad (18)$$

Fig. 6 presents absolute error values δ_{k_1} of factor estimations $k_{S_1 S_2 R}(0.99)$ for two number of degrees of freedom $m_1 = m_2 = 3$ and $m_1 = m_2 = 9$. Broken line shows characteristics of absolute error values δ_{k_2} of factor estimations $k_{SR}(0.99)$ for $m = 3$ and $m = 9$.

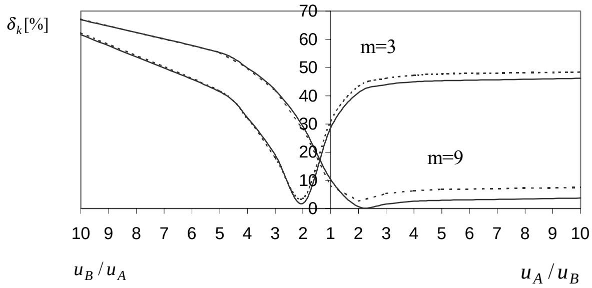

Fig. 6 Absolute error values δ_{k_1} and δ_{k_2} of coverage factor estimations in the function η and its converse

7. CONCLUSIONS

Basing on the method's accuracy analysis, the scope of application on the approximate methods of the expanded uncertainty estimation has been presented; the criterion of not-exceeding preset value of the error has been implemented. Presented characteristics of errors considered of measurement methods permit to formulate certain conclusions. As the experiments proved – ranges of applying of presented methods are different and seem to be mutually complementary in many cases.

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