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Comparison of design formulas for torsion based catapults

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Abstract **– This paper analyzes the design of ancient catapults and compares the two known design formulas for Greek catapults, based on a standardized design for Hellenistic torsion-based catapults. It is hypothesized that, as both formulas, one for an arrow shooting catapult and one for a stone thrower, were considered to give the optimum design regarding performance, that both express the same optimal design. This could be used to determine the length/width of catapult arrows for optimally designed catapults, where no catapult arrow parts so far are known to have been discovered. In order to investigate this, a set of catapult point data, known from literature, were analyzed and a mathematical model developed.**

I. INTRODUCTION

The purpose of this article is to compare the known design formulas for ancient Greek catapults regarding energy storage and optimal design. The use of catapults can have both psychological (intimidation, deterrence) as well as physical effects like targeting soldiers (small catapults) or the destruction of walls (large catapults). The physical purpose of a catapult is to move a given mass m over a certain distance d by adding a high initial velocity v to said mass m during launch from the catapult. This projectile mass m can have different shapes and forms, from arrows over irregular formed objects like natural stones to processed spherical stone balls of specific mass. But all obey the laws of physics and must carry less kinetic energy $E_{kinetic}$ than the stored energy in the torsion springs E_{launch} see below.

$$E_{kinetic} = \frac{1}{2}mv^2 < E_{launch} \tag{1}$$

The main literature regarding the standardized design of torsion based catapults (where energy is mainly stored in torsion springs) specific about two sets of standardized design rules, being compiled primarily by Marsden [1,2] and covering arrows/spears/bolts and spherical stone balls, each with its own calibration formulae to be applied, see eq. 2 and eq. 3. This is known from a section of Philon's BELOPOEICA: "*Now it is time to explain ..... the subject of artillery construction, called engine-construction by some people… the fundamental basis and unit of measure for the construction of engines was the diameter of the hole. This had to be obtained not by chance or at random, but by a standard method which could produce the correct proportions at all sizes (of a catapult). …… Later engineers looked exclusively for a standard factor with subsequent experiments as a guide……"* [1, p108-109].

This standard factor is given by the diameter of the torsion spring.

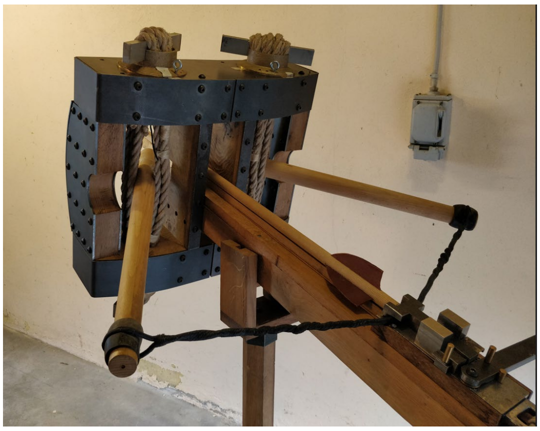

Fig. 1. Torsion based euthytonon type catapult, based on the Ampurias design, seen from the rear. Construction by author k. Paasch.

Figure 1 show a reconstructed arrow launching catapult, based on the Ampurias design with pre-tensioned rope springs inserted and is similar to the version build by Erwin Schramm in 2018. See [4] for details.

II. DESIGN METHODS

The catapult is believed to have been invented around year 399 BC in Syracuse and the design of torsion based late Hellenistic torsion-based catapults is considered to have been standardized around 220 BC (see [1] for a detailed historical description and [3] for further discussion). The expression of the measurements of all major components in a torsion-based catapult are expressed as a factor relative to a single design parameter, the diameter of the torsion spring f. This diameter is, according to the ancient sources Philon/Heron/Vitruvius [1,2], determined by two different methods, depending on the purpose of the catapult. The key factor is the diameter of the washer, holding the inserted torsion spring. For an arrow launcher (Euthytonon) the factor f_e is given by 1/9 of the actual arrow length L (eq. 2). The remains of the remains a Greek catapult, the catapult from Ampurias, Spain, is shown in figure 2. The 4 circular washers for mounting the torsion springs are still visible.

$$f_e = \frac{1}{9}L \tag{2}$$

Fig. 2. Frame of the Ampurias catapult, Museo Archeologico, Barcelona, Spain. Left: Full frame. Right: Close-up on bronze washer for torsion spring mounting. Photos by author K. Paasch.

For a stone throwing catapult (palintonon) is the washer diameter f_p calculated as only a function of the weight w of the stone (eq. 3). [1,5]. The expression is shown in SI-unit (kg for the mass m and meter for the diameter f_p). The original design formula used the unit of Attic mina.

$$f_p = 0.130\sqrt[3]{w} \tag{3}$$

The torsion springs were installed under extreme stress to ensure sufficient energy storage and individually adjusted by turning the washers. For details see [1,3,5]. The actual amount of energy stored in a torsion spring / washer design as shown in figure 1 and 2 is traditionally calculated on the basis of a solid cylinder approximation. However, recent research has shown that the missing material below the crossbar should be taken into account [6,7,8].

III. HYPHOTESIS

The physical principle behind the function of the catapults is that both types of catapults accelerate a given mass m to a given launch velocity v with an initial energy given in eq. 1. In case both design formulas express a maximum performance is it to be considered if both equations express the same function of projectile weight m. In that case should it be possible to establish the actual length and shaft diameter L and d of catapult arrows/spears/bolts, where only the points have survived [9,10], as well as determining the size of arrows for catapult parts found without corresponding arrows/bolts.

IV. COMPARISON OF FORMULAS

The "standardization" in design of catapults is as stated considered to have taken place around 220 BC. Both formulas relate to the weight of the projectile, the palintonon formula with direct use of the mass m of the stone and the euthytonon indirectly by the length L of the arrow. The mass of an arrow as function of its length however must be determined on the basis of the physical dimensions and material properties, such as specific gravity of iron and wood. This will be modelled. Under the initial hypothesis that both calibration formulas (eq. 2 and 3) express the same physical performance, we have

$$f_e = f_p \tag{4}$$

giving

$$\frac{1}{9}L = 0.130\sqrt[3]{m} \tag{5}$$

Under the hypothesis that both calibration formulas are optimized regarding the resulting mass of the projectile (stone/arrow/spear), will both formulas for a given projectile mass m give the optimal torsion spring diameter f. Combining the calibration formulas and solving for the mass m (in kg) gives

$$m = \frac{L^3}{(0.130 \cdot 9)^3} \tag{6}$$

The projectile mass m of an arrow is expressed as a function of the arrow length L and its other physical dimensions, shape and material compositions.

The full length L can further be divided into the shaft length L_{shaft} and the point length L_{point}, as illustrated in figure 3.

$$L = L_{shaft} + L_{Point} \tag{7}$$

L_{shaft} can be expressed as the ratio α between the shaft length and the full length L, for $\alpha<1$.

$$L_{shaft} = \alpha L \tag{8}$$

The mass m of an arrow can be calculated as the sum of the mass of the visible shaft (m_{shaft}) and the mass of the point section (m_{point}).

$$m = m_{shaft} + m_{point} \tag{9}$$

The mass of the shaft part m_{shaft} is calculated via its volume and the specific gravity of the wooden shaft material (δ_{shaft}) and the length L_{shaft}. The mass of the fletching is considered much lower than the other components and is not included. The real specific gravity of the point area will depend on the weight m as well of the shape of the actual arrowpoint, which might vary from type to type. As the arrowpoint can have a multitude of shapes a virtual specific gravity $\delta_{virtual}$ is introduced into the model, expressing the specific gravity in case the arrow point section was an enclosing cylinder of a pseudo-material with a virtual density $\delta_{virtual}$, resulting in the same mass m as the arrow point section as illustrated in figure 3. See section IV for details. The mass of the arrowpoint m_{point} is thus calculated via the enclosing cylinder volume and the virtual specific gravity ($\delta_{virtual}$). In this simple model is the small cone of wood inside the socket not included as, for example, an inner diameter of 16 mm and length of 70 mm inside will add around 3 gram to the weight, only few percent of the weight of the iron point itself.

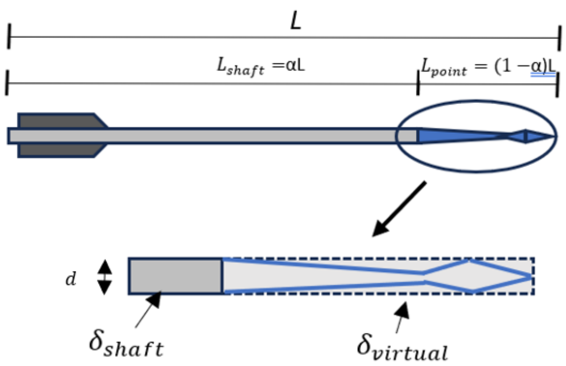

Fig. 3. Arrow composition and measurements.

The mass of the parts are approximated by

$$m_{shaft} = \delta_{shaft}\pi\frac{d^2}{4}L_{shaft} \quad (10)$$

$$m_{point} = \delta_{virtual}\pi\frac{d^2}{4}L_{point} \quad (11)$$

Inserting eq. 10 and 11 into eq. 9 gives

$$m = \frac{\pi}{4}d^2 \cdot (\delta_{shaft}L_{shaft} + \delta_{virtual}L_{point}) \quad (12)$$

resulting in

$$k_1L^3 = k_2d^2 \cdot (\delta_{shaft}L_{shaft} + \delta_{virtual}L_{point}) \quad (13)$$

where the constants k1 and k2 below contains the numerical values.

$$k_1 = \frac{1}{(0.130 \cdot 9)^3}$$

$$k_2 = \frac{\pi}{4}$$

By applying the ratio α from eq. 7 as well as eq. 13 can it be shown that the diameter d of the arrow shaft be calculated by solving the following equation

$$d^2 = \frac{k_1}{k_2} \cdot \frac{L^3}{\delta_{shaft}\alpha L + \delta_{virtual}(L - \alpha L)} \quad (14)$$

Solving for the shaft diameter d

$$d = \sqrt{\frac{4}{\pi \cdot (0.130 \cdot 9)^3}} \cdot \frac{L}{\sqrt{\delta_{shaft}\alpha + \delta_{point}(1-\alpha)}} \quad (15)$$

Rearranging for full arrow length L gives

$$L = d \cdot \sqrt{\frac{\pi \cdot (0.130 \cdot 9)^3(\delta_{shaft}\alpha + \delta_{virtual}(1-\alpha))}{4}} \quad (16)$$

The parameters L_{point} and d are measured values from the given arrowpoint under investigation, as shown in figure 4.

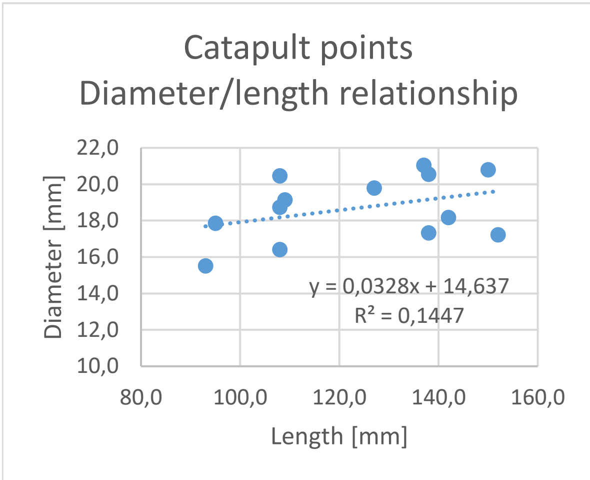

Fig. 4. Measured length versus diameter of 13 catapult points from [9, plate 14].

By introducing the variable β as the length of the bolt point L_{point} divided by the diameter d, the arrow length L can be determined.

$$\beta = \frac{L_{point}}{d} \rightarrow L_{point} = \beta d \quad (17)$$

Inserting eq. (17) into eq. (14) and rearranging terms gives

$$d^2 = \frac{k_1}{k_2} \cdot \frac{L^3}{\delta_{shaft}L_{shaft} + \delta_{point}(L - L_{shaft})}$$

$$d^2 = \frac{k_1}{k_2} \cdot \frac{L^3}{\delta_{shaft}(L - L_{point}) + \delta_{point}L_{point}}$$

$$d^2 = \frac{k_1}{k_2} \cdot \frac{L^3}{\delta_{shaft}L - \delta_{shaft}L_{point} + \delta_{point}L_{point}}$$

$$d^2 = \frac{k_1}{k_2} \cdot \frac{L^3}{\delta_{shaft}L - \delta_{shaft}\beta d + \delta_{point}\beta d}$$

$$d^2 = \frac{k_1}{k_2} \cdot \frac{L^3}{\delta_{shaft}L + \beta d(\delta_{point} - \delta_{shaft})} \quad (18)$$

It can be observed that the only unknown in eq. 18 is the arrow length L. Rearranging for powers of L:

$$\beta d^3(\delta_{point} - \delta_{shaft}) = \frac{k_1}{k_2}L^3 - d^2\delta_{shaft}L$$

$$\frac{k_1}{k_2}L^3 - d^2\delta_{shaft}L - \beta d^3(\delta_{point} - \delta_{shaft}) = 0$$

$$L^3 - \frac{k_2}{k_1}d^2\delta_{shaft}L - \frac{k_2}{k_1}\beta d^3(\delta_{point} - \delta_{shaft}) = 0 \quad (19)$$

Equation 19 shows that the length of the projectile can be calculated via 4 parameters:

- The length of the arrow point L_{point}
- The socket diameter d.
- The virtual specific gravity of the point δ_{point}.
- The specific gravity of the wooden shaft δ_{shaft}.

The calculated corresponding β-values are illustrated in figure 5. The specific gravities must be estimated.

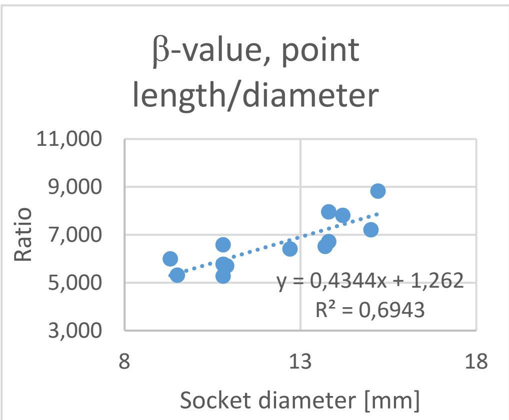

Fig. 5. Calculated point β-values as function of diameter size of the 13 catapult points from [9, plate 14].

The wood used for the shaft of roman spears throwing weapons such as pilum is expected to be a type of hardwood, typically ash, hazelnut or similar [11] and similar type of wood is due to lack of better knowledge here expected to be used for arrows. The specific gravity of these types of hardwood are typically around 670-740 kg/m^3 [12,13]. The value will also depend on the humidity content. The so far only known complete catapult arrows discovered (Dura-Euporos, Qasir Ibrim) [14] are from later Roman periods and thus not representative for this analysis.

V. VIRTUAL DENSITY OF THE ARROWPOINT

The virtual specific gravity δ_{point} of the point will depend on the detailed shape and the wood inside the socked structure. A detailed analysis has in this paper been performed on data from a find of Roman Republican weapon hoard from Grad near Šmihel under Nanos Mt. in Slovenia [9], covering a large number of socketed catapult points of various sizes. The analysis determined the virtual specific gravity of the catapult points together with an estimation of their mass, for a structure illustrated in figure 6. The virtual specific gravity of catapult points can easily be calculated, in case there is only little corrosion.

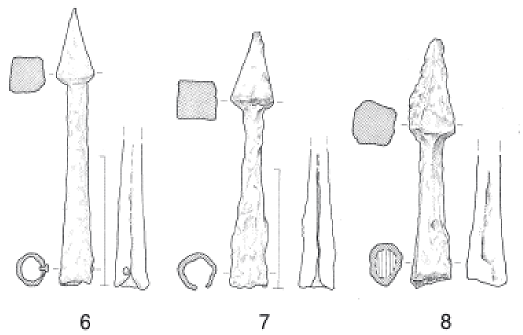

Fig. 6. Example of catapult points from the Grad near Šmihel find, Slovakia [9, plate 14]. Reproduced by kind permission from Arheološki vestnik.

A graphical representation of 13 calculated values, based on the drawings from [9, plate 14], is shown in fig. 7.

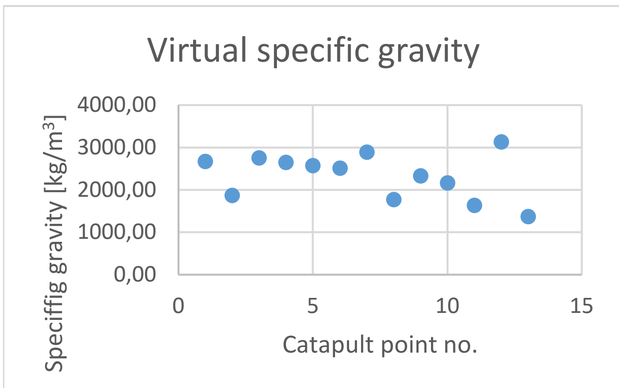

Fig. 7. Virtual specific gravity values for 13 analyzed catapult points data. Compiled from [9, plate 14].

For corroded arrowpoints is it recommended that the measured/approximated geometries of the points. Drawings of geometries of the catapult points published in [8, plate 14] were used in this study. Swelling caused by corrosion can make the estimate difficult. The average value is 2330 kg/m^3. The difference between the recorded weight of the catapult points and the estimated weight, based on the geometry and the specific gravity of iron (7800 kg/m^3) is for 7 points within 10%. Those points are used as examples i section VII.

VI. SOLUTION TO EQUATION

A depressed cubic equation like eq. (19) can be written in the general form

$$L^3 + pL + q = 0 \tag{20}$$

where in this case

$$p = -\frac{k_2}{k_1} d^2 \delta_{shaft} \tag{21}$$

$$q = -\frac{k_2}{k_1} \beta d^3 (\delta_{point} - \delta_{shaft})$$

$$= -\frac{k_2}{k_1} L_{point}\, d^2 (\delta_{point} - \delta_{shaft}) \tag{22}$$

Cubic equations in general cannot easily be solved analytically. Except in cases, where one root is obvious and the equation is reduced to a quadratic equation, cubic equations are in practice mostly solved by an approximate or numerical method. The method used here is based on Cardano's method and a set of assumptions related to its use. In case p and q are real numbers

$$\Delta = \frac{q^2}{4} + \frac{p^3}{27} > 0 \tag{23}$$

then the real root (in this case L) is given by

$$L = \sqrt[3]{u_1} + \sqrt[3]{u_2} \tag{24}$$

where

$$u_1 = -\frac{q}{2} + \sqrt{\Delta} \tag{25}$$

$$u_2 = -\frac{q}{2} - \sqrt{\Delta} \tag{26}$$

For details see [15]. Eq. 21-26 are easily implemented in software such as Matlab, Excell spreadsheet etc.. Figure 8 shown the calculated proposed arrow length L as function of the point length L, for selected values of the socket/shaft diameter (15-26 mm) for a selected specific gravity of the wooden shaft (700 kg/m^3) and selected virtual specific gravity (2400 kg/m^3) of for the point section (based on iron).

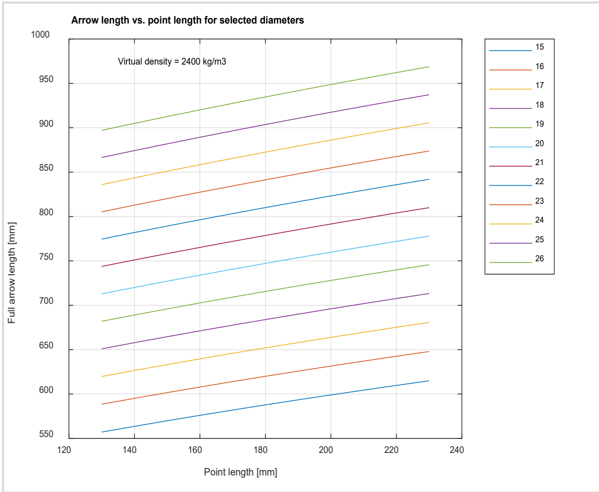

Fig. 8. Estimated full arrow length L as function of the point length L_{point} for socket diameters 15-26 mm and a point virtual density of 2400 kg/m^3)

A sensitivity analysis has been performed around the selected parameter set of d = 20 mm, L_{point} = 180 mm and δ_{point} = 2400 kg/m^3. The results are shown in table 1.

Table 1 Sensitivity analysis.

Parameter	Point of analysis	L variation
d	20 mm	3.2 cm / mm
L_{point}	180 mm	0.6 cm / cm
δ_{point}	2400 kg/m^3	0.7 cm / 100kg/m^3

The sensitivity analysis shows that especially the estimation of the socked diameter d is important, as even mm variations can lead the deviations in the cm-range for the estimation of the full arrow length L.

VII. EXAMPLES

As stated in section V, out of the 13 data sets used from [9, plate 14], 7 of them show a difference below 10% between the calculated and the measured weight. See table 2. These 7 samples are used to calculate the expected corresponding full-length L of the arrow, based on eq. 21-26.

Table 2 Calculated full arrow length.

Figure in [9, plate 14]	Length [mm]	Socked [mm]	Weight [gram]	Point sp. gravity [kg/m^3]	L [cm]
No. 1	152	17.2	93	2670	65.6
No. 2	150	20.8	92	1870	71.7
No. 6	138	17.3	84	2650	64.8
No. 7	127	19.8	98	2580	71.4
No. 10	138	20.5	112	2510	74.1
No. 12	108	20.5	62	1770	67.8
No. 15	108	18.7	63	1370	60.1

The examples in table 2 shows that for the arrow points used in the analysis, full arrow lengths in the range of 60-74 cm are to be expected. But although there are 2 different design formulas, for euthytonons and palintonons respectively, does it seem quite straightforward that arrow launchers may have been used also for launching small stones (likely in the same weight range of the correct stone size), just as palintonons were used for throwing arrows/spears [3], depending on the actual situation.

VIII. CONCLUSION

This paper investigated the possibility of combining the two known basic formulas for the diameter of the torsion springs for euthytonon and palintonon catapults. The hypothesis is based on the authors assumption that both formulas represent the same basic principle, to accelerate a given mass to an optimized velocity. The combination of both formulas and the introduction of a virtual specific gravity δ_{point} for the point section of the arrowhas led to a polynomium of 3rd degree, solvable with a real root under certain assumptions. As an example, the published geometrical forms of 7 socketed catapult points have been analyzed and used as input data.

It has been shown that the combination of the apparently different calibration formulas for standardized torsion-based catapults could possibly be used to approximate the length of an arrow for an early Hellenistic euthytonon arrow shooting catapult, designed according to the standardized design method developed around 220 BC.

To the authors knowledge no full-length catapult arrows from the Hellenistic period have been found or identified as such. Therefore, it has not been possible to test the developed theory against archaeological finds.

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