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# Sine-Wave Amplitude Estimation by a Two-Point Interpolated DFT Method Robust to Spectral Interference from the Image Component

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***Abstract* – This paper proposes a new two-point Interpolated Discrete Fourier Transform (IpDFT) method for the amplitude estimation of a sine-wave with high rejection capability of the spectral interference from the image component. The method is based on the Maximum Sidelobe Decay (MSD) windows. The analytical expression of both the proposed amplitude estimator and its variance due to additive white noise are derived. Computer simulations and experimental results show that the proposed estimator outperforms both the classical IpDFT and the three-point IpDFT methods when the number of acquired sine-wave cycles is small.**

## I. INTRODUCTION

Sine-waves are employed in a large variety of engineering applications. The Discrete Fourier Transform (DFT)-based methods are often employed to estimate their parameters since they require a small processing effort, are simple to understand and implement and provide accurate results [1]. One of the most common DFT-based methods is the so-called Interpolated DFT (IpDFT) method [2]-[4]. By this method the spectral leakage and the picket-fence effects on parameter estimates are greatly reduced by using windowing and interpolating suitably selected spectrum samples, respectively. At first the IpDFT method estimates the sine-wave frequency by determining the ratio of the two largest samples of the sine-wave spectrum. Then, the sine-wave amplitude and the phase are evaluated using the estimated frequency. Simple analytical expressions for the IpDFT parameter estimators are achieved when the Maximum Sidelobe Decay (MSD) windows are adopted [2], [4]. Unfortunately, the parameter estimates returned by the IpDFT method are affected by spectral interference from the image component, whose effect can be relevant when the number of acquired sine-waves cycles is small. In this case, multi-point IpDFT methods were proposed to reduce the detrimental effect of the image component by using more than two interpolation points [5], [6].

Recently a three-point sine-wave frequency IpDFT estimator based on the MSD windows was proposed [7] in which two of the spectral samples involved in the interpolation are weighted by suitable coefficients to compensate the contribution of the image component. It has been shown that this method returns more accurate results than the existent multi-point IpDFT methods.

The aim of this paper is to propose a two-point sine-wave amplitude IpDFT estimator based on a similar idea. It is based on the frequency estimator derived in [7] and allows the rejection of the contribution of the image component on the estimated parameter. The analytical expressions of both the proposed amplitude estimator and of its variance due to additive white noise are derived. Moreover, the accuracy of the proposed estimator is compared with those provided by the classical IpDFT and the three-point IpDFT methods by means of both computer simulations and experimental results.

## II. THE PROPOSED AMPLITUDE ESTIMATOR

Let us consider the following discrete-time sine-wave, characterized by amplitude $A$, frequency $f$, and initial phase $\varphi$, achieved from a continuous-time sine-wave using a sampling frequency $f_s$:

$$x(m) = A\sin(2\pi f m + \varphi), \qquad m = 0,1,2,\ldots \tag{1}$$

If $M$ samples are acquired, the discrete frequency $f$, can be expressed as:

$$f = \frac{f_{in}}{f_s} = \frac{\nu}{M} = \frac{l+\delta}{M}, \tag{2}$$

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where $f_{in}$ is the continuous-time signal frequency, $l$ and $\delta$ ($-0.5 \le \delta < 0.5$) are respectively the integer and the fractional parts of the acquired sine-wave cycles, $\nu$. It is worth noticing that $\nu$ represents also the discrete frequency expressed in bins. In practice non-coherent sampling is often encountered, i.e. $\delta \neq 0$.

The proposed two-point IpDFT method is based on the $H$-term MSD window ($H \ge 2$). These windows are used since they allow us to derive analytical expressions for the amplitude estimator and its variance. In addition, the $H$-term MSD window has the most rapidly decaying rate, equal to $6(2H - 1)$ dB/octave, among the $H$-term cosine windows [8]. This feature ensures a high rejection capability of the spectral interference from tones when $\nu$ is high enough. The $H$-term MSD window is defined as [4]:

$$w(m) = \sum_{h=0}^{H-1} (-1)^h a_h \cos\left(2\pi h \frac{m}{M}\right), \quad m = 0,1,\ldots,M-1 \quad (3)$$

where the window coefficients $a_h$ are given by $a_0 = \frac{C_{2H-2}^{H-1}}{2^{2H-2}}$, $a_h = \frac{C_{2H-2}^{H-h-1}}{2^{2H-3}}$, $h = 1,2,\ldots,H-1$, in which $C_p^q = p!/((p-q)!q!)$.

The DFT of the windowed signal $x_w(m) = x(m)\cdot w(m)$ can be expressed as:

$$X_w(k) = \frac{A}{2j}\left[W(k-\nu)e^{j\varphi} - W(k+\nu)e^{-j\varphi}\right], \quad k = 0,1,\ldots,M-1 \quad (4)$$

where $W(\cdot)$ is the Discrete-Time Fourier Transform (DTFT) of the window $w(\cdot)$. It can be expressed as [4]:

$$W(\lambda) = \frac{M\sin(\pi\lambda)}{2^{2H-2}\pi\lambda} \frac{(2H-2)!}{\prod_{h=1}^{H-1}\left(h^2 - \lambda^2\right)} e^{-j\pi\lambda}, \quad \text{for } |\lambda| << M \quad (5)$$

From (5) it can be deduced that $|W(\cdot)|$ is an even function, i.e., $|W(-\lambda)| = |W(\lambda)|$.
The second term in (4) represents the effect of the image component.

The proposed method employs the following DFT spectrum samples [7]:

$$\begin{aligned} |X_w(l-1)| &\cong \frac{A}{2}|W(1+\delta)| - p\frac{A}{2}|W(2l-1+\delta)|, \\ |X_w(l)| &\cong \frac{A}{2}|W(\delta)| + p\frac{A}{2}|W(2l+\delta)|, \\ |X_w(l+1)| &\cong \frac{A}{2}|W(1-\delta)| - p\frac{A}{2}|W(2l+1+\delta)|, \end{aligned} \quad (6)$$

where $p = (-1)^H \operatorname{sgn}(\delta)\cos(2\pi\delta + 2\varphi)$.
When the frequency signal-to-noise ratio is above 18-20 dB the integer part $l$ can be easily determined by applying a maximum search procedure to the DFT spectral samples $|X_w(k)|$, $k = 1, 2,\ldots, M/2 - 1$.

Two situations may occur:
A) $|X_w(l-1)| > |X_w(l+1)|$, which occurs when $-0.5 \le \delta < 0$
Using (6) we achieve:

$$\begin{aligned} |X_w(l)| + a|X_w(l-1)| &\cong \frac{A}{2}\left[|W(\delta)| + a|W(1+\delta)|\right] \\ &+ p\frac{A}{2}\left[|W(2l+\delta)| - a|W(2l-1+\delta)|\right]. \end{aligned} \quad (7)$$

This expression shows that the contribution of the image component on the spectrum can be removed if the second term in the right side is null, i.e., when:

$$a = \frac{|W(2l+\delta)|}{|W(2l-1+\delta)|}. \quad (8)$$

Using (6) we have [7]:

$$a = \frac{2l - H + \delta}{2l + H + \delta - 1}, \quad (9)$$

and

$$|W(1+\delta)| = \frac{H-1-\delta}{H+\delta}|W(\delta)|. \quad (10)$$

By replacing (9) and (10) into (7) we achieve:

$$A \cong \frac{H+\delta}{(2H-1)(l+\delta)} \times \frac{(2l+\delta+H-1)|X_w(l)| + (2l+\delta-H)|X_w(l-1)|}{|W(\delta)|}. \quad (11)$$

B) $|X_w(l+1)| > |X_w(l-1)|$, which occurs when $0 \le \delta < 0.5$
Using (6) we achieve:

$$\begin{aligned} |X_w(l)| + b|X_w(l+1)| &\cong \frac{A}{2}\left[|W(\delta)| + b|W(1-\delta)|\right] \\ &+ p\frac{A}{2}\left[|W(2l+\delta)| - b|W(2l+1+\delta)|\right], \end{aligned} \quad (12)$$

where to remove the contribution from the image component $b$ should be equal to:

$$b = \frac{|W(2l+\delta)|}{|W(2l+1+\delta)|}. \quad (13)$$

Using (6) we have [7]:

$$b = \frac{2l + H + \delta}{2l - H + \delta + 1}, \tag{14}$$

and

$$\left|W(1-\delta)\right| = \frac{H - 1 + \delta}{H - \delta}\left|W(\delta)\right|. \tag{15}$$

By replacing (14) and (15) into (12) we achieve:

$$A \cong \frac{H-\delta}{(2H-1)(l+\delta)} \times \frac{(2l+\delta-H+1)\left|X_w(l)\right| + (2l+\delta+H)\left|X_w(l+1)\right|}{\left|W(\delta)\right|}. \tag{16}$$

From (11) and (16) the following amplitude estimator can be derived:

$$\hat{A} = \frac{H+(-1)^s\hat{\delta}}{(2H-1)(l+\hat{\delta})} \times \left[\frac{\left(2l+\hat{\delta}+(-1)^s(H-1)\right)\left|X_w(l)\right|}{\left|W(\hat{\delta})\right|} + \frac{\left(2l+\hat{\delta}+(-1)^{s+1}H\right)\left|X_w\left(l+(-1)^{s+1}\right)\right|}{\left|W(\hat{\delta})\right|}\right], \tag{17}$$

where $s = 0$ if $|X_w(l-1)| > |X_w(l+1)|$, and $s = 1$ if $|X_w(l+1)| > |X_w(l-1)|$ and $\hat{\delta}$ is the frequency estimator proposed in [7].

## III. INFLUENCE OF WIDEBAND NOISE

To model real-life situations, the signal (1) is corrupted by additive white Gaussian noise with zero mean and standard deviation $\sigma_n^2$. To determine the contribution of wideband noise on the estimated amplitude, the number of acquired sine-wave cycle $\nu$ is assumed to be high enough that the contribution of the image component can be neglected. Thus, we have $a \cong 1$ and $b \cong 1$ and from (7), (10) and (12), (15) we obtain:

$$\hat{A} \cong V\left(\left|X_w(l)\right| + \left|X_w\left(l+(-1)^{s+1}\right)\right|\right), \tag{18}$$

where

$$V = \frac{2(H+(-1)^s\hat{\delta})}{(2H-1)\left|W(\delta)\right|}. \tag{19}$$

Simulations showed that the estimation bias due to noise is negligible. Moreover, by applying the uncertainty propagation law [9] to (18) we achieve:

$$\sigma_{\hat{A}}^2 \cong 2V^2(1+\rho_1)\sigma_{X_w}^2 + \left(\frac{A}{V}\right)^2\left(\frac{\partial V}{\partial \delta}\right)^2\sigma_{\hat{\delta}}^2 + 2A\frac{\partial V}{\partial \delta}\left[\rho\left(\left|X_w(l)\right|,\hat{\delta}\right) + \rho\left(\left|X_w(l+(-1)^{s+1})\right|,\hat{\delta}\right)\right]\sigma_{X_w}\sigma_{\hat{\delta}}, \tag{20}$$

where $\sigma_{X_w}^2 = \frac{M}{2}\cdot NNPG \cdot \sigma_n^2$, is the variance of the DFT spectral samples [10], in which *NNPG* is the window Normalized Noise Power Gain, equal to $NNPG = a_0^2 + 0.5\sum_{h=1}^{H-1} a_h^2 = C_{4H-4}^{2H-2}/2^{4H-4}$ [4].

Moreover, $\sigma_{\hat{\delta}}^2$ is the variance of the $\hat{\delta}$ estimator proposed in [7], which for enough high value of $l$ is given by [6]:

$$\sigma_{\hat{\delta}}^2 \cong \frac{4\pi^2\delta^2}{A^2\sin^2(\pi\delta)}\frac{\prod_{h=1}^{H}\left(h^2-\delta^2\right)^2}{\left[(2H)!\right]^2} \times \frac{C_{4H-4}^{2H-2}}{M}\frac{(4H-3)\left[(4H-1)\delta^2+H^2\right]}{H(2H-1)}\sigma_n^2 \tag{21}$$

Also, $\rho_1 = (2H-2)/(2H-1)$ is the correlation coefficient between two neighbouring DFT spectral samples $|X_w(j)|$ and $|X_w(j+1)|$ [4].

Finally, $\rho\left(\left|X_w(l-1)\right|,\hat{\delta}\right)$, $\rho\left(\left|X_w(l)\right|,\hat{\delta}\right)$, and $\rho\left(\left|X_w(l+1)\right|,\hat{\delta}\right)$ are the correlation coefficients between the spectral samples $|X_w(l-1)|$, $|X_w(l)|$, and $|X_w(l+1)|$ and $\hat{\delta}$, whose expressions are derived in [11].

If $\delta$ is not too close to zero, (20) can be accurately approximated by:

$$\sigma_{\hat{A}}^2 \cong 2V^2(1+\rho_1)\sigma_{X_w}^2. \tag{22}$$

## IV. SIMULATION AND EXPERIMENTAL RESULTS

This Section is aimed at comparing the performance of the proposed amplitude estimator with those provided by the classical IpDFT method and the three-point IpDFT method [5], [6] through both computer simulations and experimental results.

In the simulation runs both pure and noisy sine-waves were considered and the two-term MSD (or Hann) window was adopted. Furthermore, the accuracies of the analytical expressions (20) and (22) for the variance of the proposed amplitude estimator were verified. The employed frequency estimates were those returned by the considered method itself. The amplitude of the simulated sine-waves was $A$ = 2 and the number of analyzed samples was $M = 1024$. An additive white Gaussian noise with zero mean and variance corresponding to a Signal-to-Noise Ratio (*SNR*) of 50 dB was considered.

Fig. 1 shows the maximum of the magnitude of the relative amplitude estimation error $|\varepsilon_A|_{\max}$ returned by all the considered methods in the case of a pure sine-wave. The number of acquired sine-wave cycles $\nu$ was varied in the range [2.01, 16) with a step of 0.04. For each value of $\nu$ the sine-wave phase $\varphi$ was varied in the range [0, 2π) rad with a step of π/50 rad.

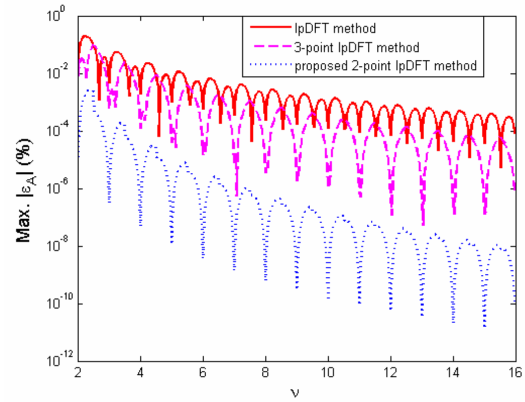



*Fig. 1. Errors $|\varepsilon_A|_{max}$ achieved by the IpDFT method, the three-point IpDFT method, and the proposed two-point IpDFT method versus $\nu$ in the case of a pure sine-wave. The Hann window is adopted and $M = 1024$.*

As we can see, the proposed method outperforms the others for all the considered values of $\nu$. The poorest accuracy is provided by the IpDFT method.

Fig. 2 shows the Root Mean Square Error (RMSE) of the considered amplitude estimators ~~provided~~ as a function of ν. For each value of ν, 1000 waveforms were generated by uniformly varying the phase $\varphi$ in the range [0, 2π) rad.

As we can see, the proposed method outperforms both the IpDFT and the three-point IpDFT methods as soon as $\nu$ is smaller than about 6. Conversely, when $\nu$ is greater than 6 all the considered methods exhibit almost the same accuracy. Also, it can be observed that the RMSE values returned by the proposed method exhibit very small variations for all the considered values of $\nu$.

Fig. 3 shows the statistical efficiency of the proposed amplitude estimator (i.e. the ratio $\left(\sigma_A^2\right)_{CR} / \left(\sigma_A^2\right)$ between the unbiased Cramer-Rao lower bound and the estimator variance) returned by simulations using 10,000 runs, (20) and (22), respectively, as a function of $\delta$. The value of $\nu$ is equal to 57 and $\delta$ was varied in the range [-0.5, 0.5) with a step of 0.04.

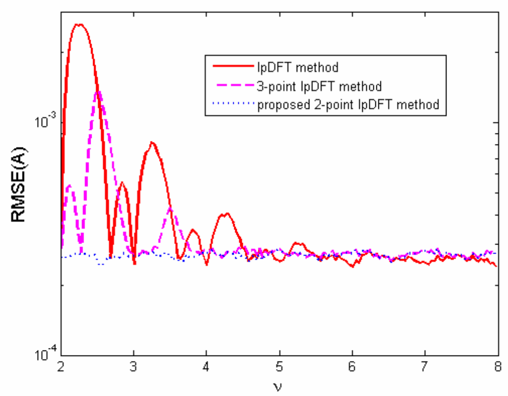



*Fig. 2. RMSE of the amplitude estimators provided by the IpDFT method, the three-point IpDFT method, and the proposed two-point IpDFT method versus $\nu$ in the case of a noisy sine-wave. The Hann window is adopted and $M = 1024$.*

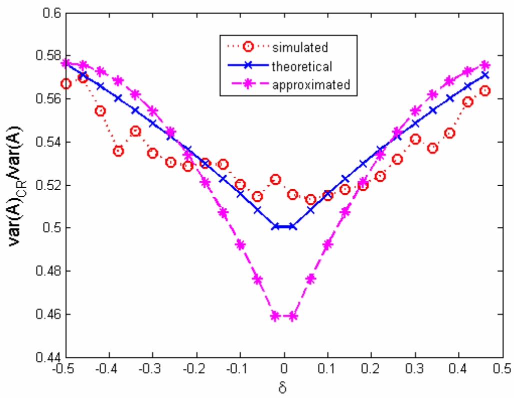


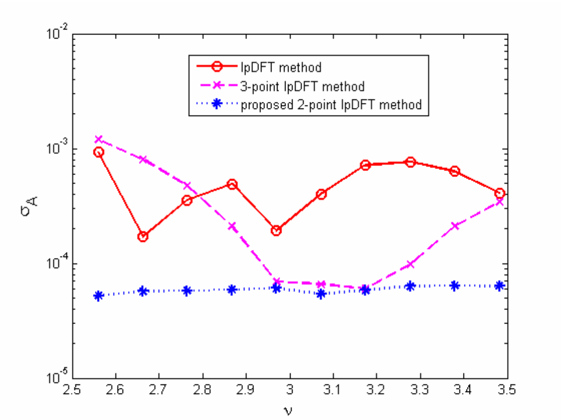

*Fig. 3. Statistical efficiency $\left(\sigma_A^2\right)_{CR} / \left(\sigma_A^2\right)$ of the proposed amplitude estimator versus $\delta$ achieved by simulation (circles and dotted line), using (20) (crosses and solid line), and using (22) (stars and dashed line). The Hann window is adopted, $\nu$ = 57, and $M = 1024$.*

Fig. 3 shows that values returned by (20) are very close to simulation results. Indeed, they differ at most by about 4.4%. Conversely, the approximated expression (22) provide accurate results when $|\delta| \geq 0.1$. Under this constraint, the maximum difference between approximated and simulated values is about 5.4% and it is reached for $\delta = -0.1$.

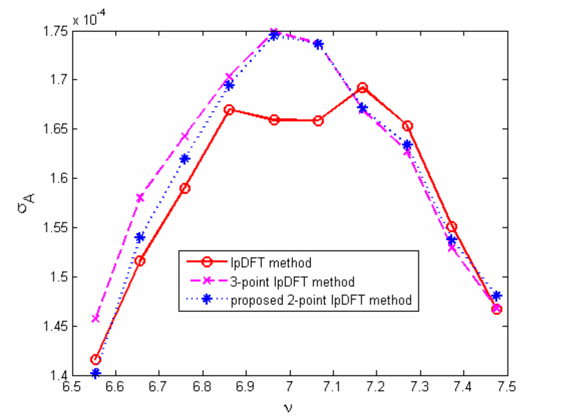
Simulations were performed using also the three-term MSD window. Behaviours very similar to those reported in Figs. 1-3 were achieved. Moreover, the same behaviours have been achieved when the noise is uniformly distributed (e.g. quantization noise).

In the experimental runs the sine-waves were generated by an Agilent 33220A signal generator and acquired by means of a 12-bit data acquisition board NI 6023. The full-scale range and the sampling frequency of the acquisition board were set to 10 V and 100 kHz, respectively. The amplitude of the sine-waves were set to

2 V, whereas the frequencies were varied in the ranges (250, 340) Hz and (640, 730) Hz, respectively, with a step of 10 Hz. The above ranges correspond to an integer part of the number of observed cycles $l$ equal to 3 and 7, respectively. The *SNR* of the analyzed signals was about 60 dB. For each frequency 1000 runs of $M$ = 1024 samples each were performed and the standard deviations of the amplitude estimators provided by the IpDFT, three-point IpDFT, and the proposed two-point IpDFT methods were determined. The Hann window was adopted. The achieved results are shown in Fig. 4 as a function of the value of $\nu$ achieved as the mean value of the estimates returned by the method proposed in [7].




(a)




(b)

*Fig. 4. Standard deviations of the amplitude estimators provided by the IpDFT, three-point IpDFT, and proposed two-point IpDFT methods versus* $\nu$ *when the input frequencies are in the range: (a) (250, 340) Hz and (b) (640, 730) Hz. The Hann window was adopted in the methods and 1000 runs of* $M$ *= 1024 samples each were used. The sine-waves were provided by an Agilent 33220A signal generator and have SNR* $\cong$ *60 dB.*

The experimental results show, in agreement with simulations, that the proposed method overcomes the others when $\nu$ = 3. Conversely, when $\nu$ = 7 the standard deviations achieved by all methods are almost the same, except the values around $\nu$ = 7 where the IpDFT method provides a little more accurate amplitude estimates.

## IV. CONCLUSIONS

In this paper a new two-point sine-wave amplitude IpDFT estimator that rejects the spectral interference from the image component has been proposed. The analytical expressions of both the estimator and its variance due to additive wideband noise have been derived. Moreover, an accurate expression for the estimator variance which holds even when the sine-wave is non-coherently sampled has been provided. It has been shown through both computer simulations and experimental results that the proposed method outperforms both the classical IpDFT method and the three-point IpDFT method in the case of pure or noisy sine-waves. Significant accuracy improvement is achieved when the number of analyzed sine-wave cycles is small.

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