A system for the calculation of uncertainty

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Abstract- The goal of this paper is to present a system to calculate uncertainties in measurement. The implementation is completed and the system is now being tested. In order to participate in the testing process, point your web browser to: http://212.51.217.114/ and login using the account GUM or GUMEK with a password of GUM. If you find any errors or have any feedback on the usability of the system, please send an email to jerzykor@p.lodz.pl. If you find any errors, please include any details or supporting information that will allow diagnosis and correction of the error.

The target of the system

The system allows calculation of expanded uncertainties for direct and indirect measurements with a large number of components, so, in practice there are no limitations The elaborated and implemented algorithms of uncertainty calculation are applied especially to randomized and centralized measurement errors.

A measurement equation can be composed of elements, which are characterized by of type A and type B error components. The system provides for the calculation of the following probability density functions of errors: rectangular, triangular, Gaussian (Normal), t-Student, arc sine (U-shaped), trapezoidal and the curvilinear trapezoidal

The level of confidence can be any value between 0 and 1, but if Gaussian or t-Student appears, then the maximum level of confidence can be p = 0,99.

The calculation of the uncertainty can be executed for any number of repeated elementary measurements (trails) with no limitation for degrees of freedom of t-Student distribution.

The error in calculation of expanded uncertainty is not bigger than 0,1% of the calculated value, and it is a compromise only to make the time of calculations reasonable. The uncertainty is rounded off to three significant digits, as well as k-factor.

The principle of calculation of expanded uncertainty

Measured quantities: both input components X_i and measurand Y – are treated as random variables, characterised by two random variables: expected value and error. The first one consists of nonzero value in general with delta Dirac distribution and the second one is a symmetrically-distributed variable with zero expected value and nonzero standard deviation.

$$X_i = \overline{X}_i + \Delta X_i \tag{1}$$

$$Y = \overline{Y} + \Delta Y \tag{2}$$

For input components, it is assumed that their expected values are the arithmetic average of elementary measurements. The errors of indirect measurements ΔX_i can be composed of several components, coming from different influencing parameters on measurements or just disturbances.

$$\Delta X_i = \sum_{j=1}^{M_i} \Delta X_{ij} \tag{3}$$

Every component has the symmetrical, centred probability density distribution $h_{ij}(x_i)$, which in many cases can be characterised by standard uncertainty $u_j(x_i)$ and distribution.

The relation between output and inputs of measurement quantity is defined by the following equation:

$$Y = f\left(X_1, X_2, \ldots X_N\right) \tag{4}$$

The expected value of the measurand is calculated based on expected values of inputs and the functional relationship between output and inputs.

$$\overline{Y} = f\left(\overline{X}_1, \overline{X}_2, \ldots \overline{X}_N\right) \tag{5}$$

The error of the measurand ΔY can be calculated using the Taylor series with partial derivatives of first order (we assume that the second and higher order partial derivatives of the Taylor series are neglectable):

$$\Delta Y = \sum_{i=1}^{N} \frac{\partial Y}{\partial X_i} \Delta X_i = \sum_{i=1}^{N} c_i \Delta X_i = \sum_{i=1}^{N} c_i \sum_{j=1}^{M_i} \Delta X_{ij} \tag{6}$$

where c_i are sensitivity factors. The probability density function of the error of the measurand $g(y)$ is a convolution of the probability density functions of errors of contribution quantities of inputs calculated to measure of output quantity.

$$h_{ij}(x_i) \Rightarrow g_{ij}(c_i x_i) = g_{ij}(y) \tag{7}$$

$$g(y) = g_{11}(y) * g_{12}(y) * \ldots * g_{NM}(y) \tag{8}$$

The integral of probability density distribution function $g(y)$, the cumulative PDF – $G(y)$, allows formulation an equation (9). Using equation (9), it is possible to assign expanded uncertainty interval at p level of confidence.

$$\int_{-U}^{U} g(y)dy = G(U) - G(-U) = p \tag{9}$$

For a symmetrical PDF, the following relation is valid:

$$U = G^{-1}\left(\frac{1+p}{2}\right) = -G^{-1}\left(\frac{1-p}{2}\right) \tag{10}$$

where $G^{-1}(\alpha)$ is an α–fractile of the $G(y)$ distribution. This relation is a base for the calculation of measurement uncertainty.

The record representing measurement result with expanded uncertainty U is worth considering. The coverage factor k, was previously used in the calculation of expanded uncertainty:

$$k = \frac{U}{u_c} \tag{11}$$

where u_c is combined standard uncertainty, defined as geometrical sum of all products of coefficients of sensitivity and standard uncertainties $c_i u_j(x_i)$. This means that components $u_{ij}(y)$ contribute in combined uncertainty:

$$u_c(y) = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{M_i} u_{ij}^2(y)} \tag{12}$$

The system assures the calculation the expanded coefficient k as well.

Procedure of calculation

Calculation of expanded uncertainty at p level of confidence requires:

- estimation of parameters of all contributing components in uncertainty;
- probability density function of errors; and
- desired level of confidence.

The parameters of the PDF are standard uncertainties $u_j(x_i)$ of type A (unbiased standard uncertainty of the mean value of measurement results – trails) or type B (standard deviation, which depends on shape of the assumed PDFs and limits of error)

The standard uncertainties are calculated as follows:

- normal distribution $u = \frac{S}{\sqrt{n}}$ (13)

- t-Student distribution $u = \sqrt{\frac{\sum_{k=1}^{n}(x_k - \bar{x})^2}{n(n-1)}}$ (14)

- rectangular distribution $u = \frac{\Delta}{\sqrt{3}}$ (15)

- triangular distribution $u = \frac{\Delta}{\sqrt{6}}$ (16)

- trapezoidal distribution $u = \sqrt{\frac{\Delta_L^2 + \Delta_H^2}{6}}$ (17)

- U-shape distribution U $u = \frac{\Delta}{\sqrt{2}}$ (18)

- rectangular distribution with logarithmic edges

$$u = \frac{1}{3}\sqrt{\Delta_L^2 + \Delta_L \Delta_H + \Delta_H^2} \quad (19)$$

where: S - the standard deviation, n - the number of elementary measurements, Δ - limit of error, Δ_L - the half of length of the bottom basis of the PDF, Δ_H - the half of length of the upper basis of the PDF. The description of a curvilinear trapezoid PDF (with logarithmic edges) follows in the Appendix.

Moreover, the sensitivity coefficients c_i, have to be calculated as the partial derivatives of the functions of measurement in relation to each component. (The new, expended version of the software does these calculations automatically).

The calculated data mentioned herein should be placed in an Excel table in one of templates, provided. The user of system has a choice of three templates: the first two are designed for a basic version of the programme, and the third an expanded version of the software (accessible for authorized users).
In the first template, one should to write all above-mentioned data. This means that the user must calculate the sensitivity coefficients by himself. If the value of measurand is placed in cell B41, then that value is automatically passed on to the table with the calculation of results.
The second template is powered by a software module, which calculates the sensitivity coefficients based on given estimates of the measured indirect quantities (components) in place of the function of measurement. The programme stress he individual measurements of standard uncertainties, coefficients of sensibility self-actively as well as components contributing to combined uncertainty.

The third template (called expanded template), is designed for wider range of PDFs, as including trapezoidal and curvilinear trapezoid (the logarithmic edges) of shaped PDFs. In this template the calculations of standard uncertainties of all contribution components are performed automatically by the software and not by user. Only the limits of errors have to be declared by the user.

All templates have incorporated explanations of how to complete the tables. Also, examples of templates containing completed examples of values are given.

Attention! Do not move tables in templates into another locations in the template! Using the “Drag” method to transfer cell values will cause a failure of Excel macro used to perform calculations. If :Drag” operation is performed, it is best to “Undo” it. It is recommended to turn of the option “permit on threading and dropping of cells” in the “Tools →Options→Edition” menu. It is possible to “Cut and paste” cell contents from other location in the template to the desire location.

After the completion of the table, the template should be saved as a text file (txt) with tab characters as delimiters and send to the computational part of the system. The first and second type of templates should send as "Basic”, and third template - as "Extended”. It is also possible to send the template without creating of text file - across text fields. In this case, the user has to mark the fragment of the template starting from cell A1 to final cell in the template (K41 or N41), copy this fragment by pressing keys Ctrl - C, go-over to the system, click into text field and insert the text by pressing keys Ctrl–V. Next, click on "Send to system”.

The calculation usually takes several seconds. If, among the PDFs are the t-Student with a small number of degrees of freedom (bellow 4), or the U-shaped PDFs, not contributing much to overall uncertainty, then calculation may take more time. Also, the time of calculation increases for higher confidence level.

As a result of calculations, the value of expanded uncertainty U and the level of coefficient k as well as two graphs are displayed.

The first graph presents the probability density function of measurement errors of all components presented in the order as they were written into table of the template.

The second graph represents convolutions of PDFs (also in order as components appear in table) and finally the distribution of all convoluted components. The components of which contribution in expanded uncertainty is not bigger that 0,005% (it means that their values are lower than 1% of maximum of the biggest contributor) are excluded from convolution calculation. Excluded contributors are marked as shaded values-lines.

Calculation of expanded uncertainty based on cumulative convolution of probability density function of all contribution uncertainties in overall uncertainty at p level of confidence does not require the value of the coverage factor k, however, due to the traditional method of calculation of uncertainties k is displayed in commentary to displayed final results.

The extended version of the software allows calculation of expanded uncertainty for probability distribution functions of which symmetry is shifted from 0y axis. Such a PDF is characterising quantisation error of a voltmeter of dual integer type. It is a PDF in range [-Δ_q;0], the symmetry axis is shifted by $\Delta_S = -\Delta_q/2$. In the extended version of software the correction k_S is displayed. k_S is calculated as the arithmetic sum of all shifts calculated to output quantity and multiplied by -1:

$$k_S = -\sum_{i=1}^{N}\sum_{j=1}^{M_i} c_i \Delta_{Sij}(x_i) \tag{20}$$

the correction k_S should be added to measurement result $\overline{Y}$. In most cases this correction does not have much influence much, so it can be ignored in calculations.

In the case of existence of correlation between results of elementary measurements the correlation two indirect measured quantities, the following procedure can be used.

If two input quantities in measurement equation X_1 and X_2 are correlated or even if there is a presumption of the existence exist of correlation, then one should count coefficient of their correlation r_{12}, using for example, the function of correlation coefficients in Excel package If the result appears in Excel appears as "# “DIVISION/0!”, the $r_{12} = 0$).
If the first template is used, one should insert in the table neither values of the standard uncertainties $u_j(x_1)$ and $u_j(x_2)$ of the type A (t-Student) of correlated quantities, nor the c_1 and c_2 sensitivity factors, but each row should contain (in suitable columns) - the following data:

- symbol of the quantity (for example u_{12});
- standard uncertainty, calculated as follows:

$$u(x_{12}) = \sqrt{[c_1 u_j(x_1)]^2 + [c_2 u_j(x_2)]^2 + 2[c_1 u_j(x_1)][c_2 u_j(x_2)]r_{12}} \tag{21}$$

- unit of standard uncertainty identical as measurand,
- t – Student distribution,

- the sensitivity coefficient c_i = 1 without unit,
- the degree of freedom equal to number of repeated elementary measurements of one of correlated quantities minus one.

Standard uncertainties of type the B of correlated quantities and their sensitivity coefficients c_1 and c_2 should be inserted in the table.
When the table is completed, then the whole template should be send to computational part of system.

If the second or third template is used: it is possible to have automatically counted components of compound uncertainty, coming from correlated inputs, and to insert these values to Equ. 21. Calculated in this way, are contribution of correlated quantities are inserted in one of the two last rows of the table, as a standard uncertainty in second template or as a standard deviation in third template. Moreover, uncertainties which concern t-Student distributions of both correlated quantities should be removed, (or zero values should be inserted in place) of their standard uncertainties or standard deviations. It is also need to write an adequate symbol in column A and an appropriate value for the degrees of freedom in column K.

The set of templates also includes an example of how to complete the table in the case of existence of the correlation between two input quantities.

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Appendix

The parameters and the shape of rectangular - logarithmic PDF.
Probability density function of such a shape is described as follows:

$$g(\xi)=0 \qquad for \qquad \xi<-\Delta_H$$

$$g(\xi)=\frac{1}{2(\Delta_L-\Delta_H)}\ln\frac{\Delta_L}{-\xi} \qquad for \qquad -\Delta_L\le\xi<-\Delta_H$$

$$g(\xi)=\frac{1}{2(\Delta_L-\Delta_H)}\ln\frac{\Delta_L}{\Delta_H} \qquad for \qquad -\Delta_H\le\xi<\Delta_H$$

$$g(\xi)=\frac{1}{2(\Delta_L-\Delta_H)}\ln\frac{\Delta_L}{\xi} \qquad for \qquad \Delta_H\le\xi<\Delta_L$$

$$g(\xi)=0 \qquad for \qquad \Delta_L\le\xi$$

Standard deviation is expressed as

$$\sigma=\frac{1}{3}\sqrt{\Delta_L{}^2+\Delta_L\Delta_H+\Delta_H{}^2}$$

If relation of lengths of bases upper to bottom is bigger than 0,6 then the pdf becomes a trapezoidal shape in practise.

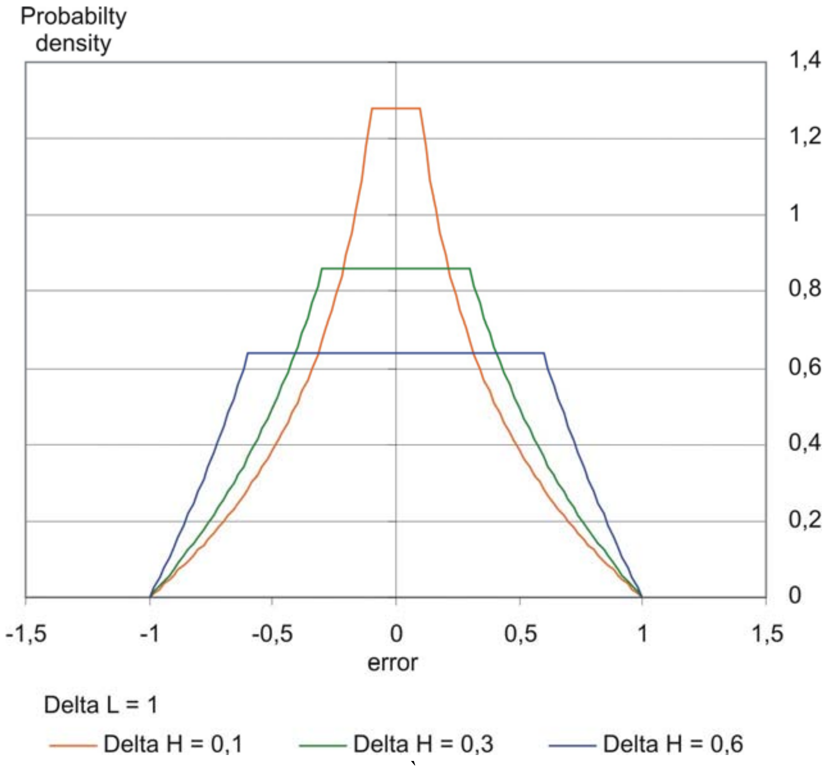

Fig. 1 The implemented shape of a curvilinear trapezoid type of PDF